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$$f(x)\cdot f'(x) = e^{2x} \Leftrightarrow$$

$$2f(x)\cdot f'(x) = 2\cdot e^{2x} \Leftrightarrow$$

$$\left(f^2(x)\right)' = \left(e^{2x}\right)' \Leftrightarrow$$

$$f^2(x) = e^{2x} + c,\quad x\in\mathbb{R}$$

για $x=0 \Rightarrow f^2(0) = e^0 + c \Leftrightarrow$

$$\Rightarrow 1 = 1 + c \Leftrightarrow c = 0$$

$$\Rightarrow f^2(x) = e^{2x} \Rightarrow \sqrt{f^2(x)} = \sqrt{e^{2x}}$$

$$\boxed{|f(x)| = e^x \quad (1)}$$

Λύνω την εξίσωση $f(x)=0 \Leftrightarrow$

$|f(x)| = 0 \overset{(1)}{\Leftrightarrow} e^x = 0$ αδύνατη.

Άρα $f(x)\neq 0\ \ \forall x\in\mathbb{R}$

f συνεχής στο $\mathbb{R}$

Από σχόλιο Bolzano

$\Rightarrow f(x)$ διατηρεί πρόσημο στο $\mathbb{R}$.

Αφού $f(0)=1>0 \Rightarrow f(x)>0$ στο $\mathbb{R}$

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$$f'(x) + x\cdot f(x) = x \quad (1)$$

$$\underbrace{f'(x)\cdot e^{\frac{x^2}{2}} + \underbrace{x\cdot e^{\frac{x^2}{2}}}_{\left(e^{x^2/2}\right)'}\cdot f(x)}_{\left(f(x)\cdot e^{x^2/2}\right)'} = x\cdot e^{\frac{x^2}{2}}$$

$$\left(f(x)\cdot e^{x^2/2}\right)' = \left(e^{x^2/2}\right)'$$

$$f(x)\cdot e^{x^2/2} = e^{x^2/2} + c$$

για $x=0$ $\quad f(0)\cdot e^0 = e^0 + c \Leftrightarrow$

$$2 = 1 + c \Leftrightarrow c = 1$$

$$f(x)\cdot e^{x^2/2} = e^{x^2/2} + 1$$

$$\boxed{f(x) = 1 + e^{-x^2/2}} \quad \forall x\in\mathbb{R}$$

$\boxed{9}$

$$f'(x)\cdot f(x) - x\cdot f^2(x) = x \Leftrightarrow$$

$$2f'(x)\cdot f(x) - 2x f^2(x) = 2x \Leftrightarrow$$

$$\left(f^2(x)\right)' - 2x\cdot f^2(x) = 2x \Leftrightarrow$$

$$\left(f^2(x)\right)'\cdot e^{-x^2} \underbrace{- 2x\cdot e^{-x^2}}_{\left(e^{-x^2}\right)'}\cdot f^2(x) = -\left(-2x\cdot e^{-x^2}\right) \Leftrightarrow$$

$$\left(f^2(x)\right)' = 2f(x)\cdot f'(x)$$

$$\left(e^{-x^2}\right)' = -2x\cdot e^{-x^2}$$

$$\left(f^2(x)\cdot e^{-x^2}\right)' = \left(-e^{-x^2}\right)' \Leftrightarrow \boxed{f^2(x)\cdot e^{-x^2} = -e^{-x^2} + c \quad (1)}$$

για $x=0 \rightarrow f^2(0)\cdot e^0 = -e^0 + c \Leftrightarrow (e-1)\cdot 1 = -1 + c \Leftrightarrow c = e$

$f^2(x)\cdot e^{-x^2} = -e^{-x^2} + e \Leftrightarrow f^2(x) = -1 + e^{x^2+1}$ (2)

εξετάζω αν $e^{x^2+1} - 1 \geq 0 \Leftrightarrow e^{x^2+1} \geq 1 \Leftrightarrow x^2+1 \geq 0$ ισχύει

$(2) \Rightarrow |f(x)| = \sqrt{e^{x^2+1} - 1}$. Λύνω την εξίσωση $f(x) = 0 \Leftrightarrow |f(x)| = 0 \Leftrightarrow$
$\sqrt{e^{x^2+1} - 1} = 0 \Leftrightarrow e^{x^2+1} = 1 \Leftrightarrow x^2+1 = 0$ αδύνατο.

Άρα $f(x) \neq 0 \ \forall x \in \mathbb{R}$
f συνεχής στο $\mathbb{R}$
$\Rightarrow$ από σχόλιο Bolzano διατηρεί πρόσημο, $f(0) > 0 \Rightarrow$
$f(x) > 0 \ \forall x \in \mathbb{R} \Rightarrow f(x) = \sqrt{e^{x^2+1} - 1}$, $x \in \mathbb{R}$.

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$\boxed{10}$

$$f'(x) = \frac{x-1}{x} \cdot f(x) \Leftrightarrow$$

$$f'(x) - \frac{x-1}{x} \cdot f(x) = 0 \Leftrightarrow$$

$$f'(x) - \left(\frac{x}{x} - \frac{1}{x}\right) \cdot f(x) = 0 \Leftrightarrow$$

$$f'(x) - \left(1 - \frac{1}{x}\right) \cdot f(x) = 0 \Leftrightarrow$$

$$f'(x) + \left(\frac{1}{x} - 1\right) \cdot f(x) = 0 \Leftrightarrow$$

$$\underbrace{f'(x) \cdot e^{\ln x - x} + \underbrace{e^{\ln x - x} \cdot \left(\frac{1}{x} - 1\right)}_{\left(e^{\ln x - x}\right)'} \cdot f(x)} = 0 \Leftrightarrow$$

$$\left(f(x) \cdot e^{\ln x - x}\right)' = 0 \Leftrightarrow f(x) \cdot e^{\ln x - x} = c \rightarrow f(1) \cdot e^{0-1} = c \Leftrightarrow$$

$$2 \cdot e^{-1} = c \Leftrightarrow \boxed{c = \frac{2}{e}}$$

$$f(x) \cdot e^{\ln x - x} = \frac{2}{e} \Leftrightarrow \boxed{f(x) = \frac{2}{e} \cdot e^{x - \ln x}}$$