Α΄ ΟΜΑΔΑΣ

3. Αν $\vec{\alpha}=(1,0)$ και $\vec{\beta}=(1,1)$, να βρείτε τον $\lambda \in \mathbf{R}$, ώστε:
(i) Τα διανύσματα $\vec{\alpha}$ και $\vec{\alpha}+\lambda\vec{\beta}$ να είναι κάθετα
(ii) Τα διανύσματα $\vec{\beta}$ και $\vec{\alpha}+\lambda\vec{\beta}$ να είναι κάθετα.

$$\vec{a}+\lambda\vec{\beta}=(1,0)+\lambda(1,1)=(1+\lambda,\lambda)$$

$$\vec{a}\perp\left(\vec{a}+\lambda\vec{\beta}\right)\Leftrightarrow\vec{a}\cdot\left(\vec{a}+\lambda\vec{\beta}\right)=0\Leftrightarrow(1,0)\cdot(1+\lambda,\lambda)=0\Leftrightarrow 1+\lambda=0\Leftrightarrow\lambda=-1$$

$$\vec{\beta}\perp\left(\vec{a}+\lambda\vec{\beta}\right)\Leftrightarrow\vec{\beta}\cdot\left(\vec{a}+\lambda\vec{\beta}\right)=0\Leftrightarrow(1,1)\cdot(1+\lambda,\lambda)=0\Leftrightarrow 1+\lambda+\lambda=0\Leftrightarrow$$
$$\Leftrightarrow 2\lambda=-1\Leftrightarrow\lambda=-\frac{1}{2}$$

7. Αν $|\vec{\alpha}| = |\vec{\beta}| = 1$ και $(\widehat{\vec{\alpha}, \vec{\beta}}) = \frac{2\pi}{3}$, να υπολογίσετε τη γωνία των διανυσμάτων $\vec{u} = 2\vec{\alpha} + 4\vec{\beta}$ και $\vec{v} = \vec{\alpha} - \vec{\beta}$.

$$\vec{a} \cdot \vec{\beta} = |\vec{a}| \cdot |\vec{\beta}| \cdot \sigma\upsilon\nu\left(\widehat{\vec{a}, \vec{\beta}}\right) = 1 \cdot 1 \cdot \sigma\upsilon\nu\frac{2\pi}{3} = -\sigma\upsilon\nu\frac{\pi}{3} = -\frac{1}{2}$$

$$\vec{u} \cdot \vec{v} = \left(2\vec{a} + 4\vec{\beta}\right)\left(\vec{a} - \vec{\beta}\right) = 2\vec{a}^2 - 2\vec{a}\vec{\beta} + 4\vec{a}\vec{\beta} - 4\vec{\beta}^2 = 2\vec{a}^2 + 2\vec{a}\vec{\beta} - 4\vec{\beta}^2 =$$
$$= 2|a|^2 + 2\alpha\beta - 4|\beta|^2 = 2 + 2\left(-\frac{1}{2}\right) - 4 = 2 - 1 - 4 = -3$$

$$|\vec{u}|^2 = \left(2\vec{a} + 4\vec{\beta}\right)^2 = 4\vec{a}^2 + 16\vec{a}\vec{\beta} + 16\vec{\beta}^2 = 4 - 8 + 16 = 12$$
$$|\vec{u}| = \sqrt{12} = 2\sqrt{3}$$

$$|\vec{v}|^2 = \left(\vec{a} - \vec{\beta}\right)^2 = \vec{a}^2 - 2\vec{a}\vec{\beta} + \vec{\beta}^2 = 1 + 1 + 1 = 3$$
$$|\vec{v}| = \sqrt{3}$$

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \sigma\upsilon\nu\left(\widehat{\vec{u}, \vec{v}}\right) \Leftrightarrow -3 = 2\sqrt{3} \cdot \sqrt{3} \cdot \sigma\upsilon\nu\left(\widehat{\vec{u}, \vec{v}}\right) \Leftrightarrow$$
$$\Leftrightarrow \sigma\upsilon\nu\left(\widehat{\vec{u}, \vec{v}}\right) = \frac{-3}{2 \cdot 3} \Leftrightarrow \sigma\upsilon\nu\left(\widehat{\vec{u}, \vec{v}}\right) = -\frac{1}{2} \Leftrightarrow \sigma\upsilon\nu\left(\widehat{\vec{u}, \vec{v}}\right) = -\sigma\upsilon\nu\frac{\pi}{3} \Leftrightarrow$$
$$\Leftrightarrow \sigma\upsilon\nu\left(\widehat{\vec{u}, \vec{v}}\right) = \sigma\upsilon\nu\frac{2\pi}{3} \Leftrightarrow \left(\widehat{\vec{u}, \vec{v}}\right) = \frac{2\pi}{3}$$

9. Να αποδείξετε ότι τα διανύσματα $\vec{u} = |\vec{\alpha}|\vec{\beta} + |\vec{\beta}|\vec{\alpha}$ και $\vec{v} = |\vec{\alpha}|\vec{\beta} - |\vec{\beta}|\vec{\alpha}$ είναι κάθετα.

$$\vec{u} \cdot \vec{v} = \left(|\vec{a}|\vec{\beta} + \left|\vec{\beta}\right|\vec{a}\right)\left(|\vec{a}|\vec{\beta} - \left|\vec{\beta}\right|\vec{a}\right) = \left(|\vec{a}|\vec{\beta}\right)^2 - \left(\left|\vec{\beta}\right|\vec{a}\right)^2 =$$

$$= |\vec{a}|^2\vec{\beta}^2 = \vec{a}^2\left|\vec{\beta}\right|^2 = \vec{a}^2\vec{\beta}^2 - \vec{\beta}^2\vec{a}^2 = 0$$

Άρα $\vec{u} \perp \vec{v}$