3.3 Reduction to the First Quadrant

In section 3.1 we saw a table with the trigonometric ratios of several basic angles. All these angles lie in the interval $[0,\frac{\pi}{2}]$, that is, in the first quadrant.

In practice, however, we often need to calculate trigonometric ratios of angles that are greater than $\frac{\pi}{2}$, or even negative. To achieve this, we use the reduction to the first quadrant: a process that allows us to transform any angle into an equivalent form that "returns" to the first quadrant, where we already know the exact trigonometric ratios.

In this way, the trigonometric ratio of a "difficult" angle is ultimately calculated using:

- an acute angle θ in the first quadrant,
- the correct sign, determined by the quadrant in which the original angle lies,
- and, in certain cases, a simple change such as ημ becoming συν, or εφ becoming σφ.

1. Angles of the form kπ ± θ (k an integer)

For angles of this form:

- The type of the trigonometric ratio does not change. It remains ημ, συν, εφ or σφ, just as in the original expression.
- The sign (+ or –) is determined by the sign of the trigonometric ratio we are calculating in the quadrant where the original angle $k\pi \pm \theta$ lies.

Examples

I. Opposite angles

Here the angles are θ and −θ=0π−θ. If θ lies in the first quadrant, then −θ lies in the fourth quadrant, where συν is positive and all the others are negative. Therefore:

$$\eta\mu(-\theta) = -\eta\mu\theta \qquad \sigma\upsilon\nu(-\theta) = \sigma\upsilon\nu\theta$$
$$\varepsilon\varphi(-\theta) = -\varepsilon\varphi\theta \qquad \sigma\varphi(-\theta) = -\sigma\varphi\theta$$

Thus, opposite angles have the same cosine and opposite values for the other trigonometric ratios.

II. Supplementary angles

Here the angles are θ and π−θ. If θ lies in the first quadrant, then π−θ lies in the second quadrant, where ημ is positive and all the others are negative. Therefore:

$$\eta\mu(\pi-\theta) = \eta\mu\theta \qquad \sigma\upsilon\nu(\pi-\theta) = -\sigma\upsilon\nu\theta$$
$$\varepsilon\varphi(\pi-\theta) = -\varepsilon\varphi\theta \qquad \sigma\varphi(\pi-\theta) = -\sigma\varphi\theta$$

Thus, supplementary angles have the same sine and opposite values for the other trigonometric ratios.

III. Angles that differ by π

Here the angles are θ and π+θ. If θ lies in the first quadrant, then π+θ lies in the third quadrant, where both ημ and συν are negative, while εφ and σφ are positive. Therefore:

$$\eta\mu(\pi+\theta) = -\eta\mu\theta \qquad \sigma\upsilon\nu(\pi+\theta) = -\sigma\upsilon\nu\theta$$
$$\varepsilon\varphi(\pi+\theta) = \varepsilon\varphi\theta \qquad \sigma\varphi(\pi+\theta) = \sigma\varphi\theta$$

2. Angles of the form $\frac{k\pi}{2} \pm \theta$ (k an odd integer)

For angles of this form:

- The type of the trigonometric ratio changes. If the original ratio is ημ, it becomes συν, and so on, according to the following correspondences: ημ ↔ συν, εϕ ↔ σϕ.
- The sign (+ or –) is determined by the sign of the original trigonometric ratio in the quadrant where the angle $\frac{k\pi}{2} \pm \theta$ lies.

Example: $\sigma\upsilon\nu(\frac{\pi}{2} + \theta) = -\eta\mu\theta$, because: συν becomes ημ when starting from $\frac{\pi}{2}$, and in the second quadrant (where $\frac{\pi}{2}+\theta$ lies), συν is negative.

In particular, we note the case of complementary angles:

IV. Complementary angles

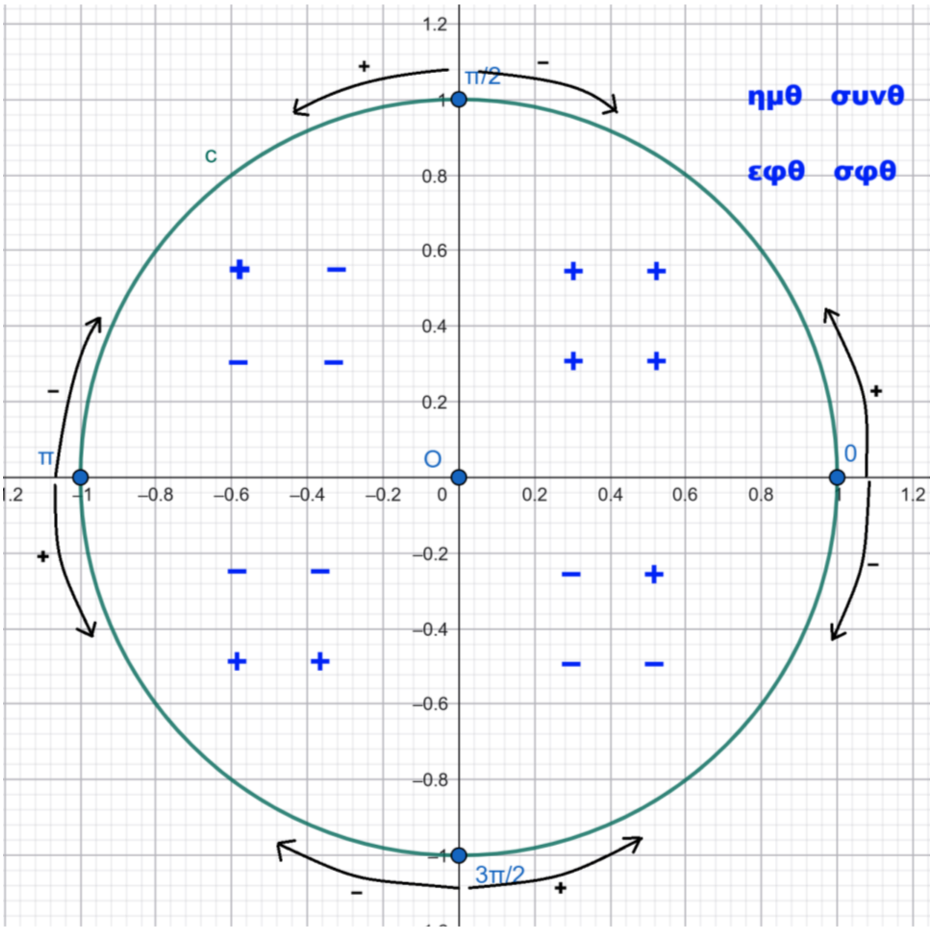
Here the angles are θ and $\frac{\pi}{2}-\theta$. If θ lies in the first quadrant, then $\frac{\pi}{2}-\theta$ is also in the first quadrant, where all trigonometric ratios are positive. Therefore:

$$\eta\mu\left(\frac{\pi}{2}-\theta\right) = \sigma\upsilon\nu\theta \qquad \sigma\upsilon\nu\left(\frac{\pi}{2}-\theta\right) = \eta\mu\theta$$
$$\varepsilon\varphi\left(\frac{\pi}{2}-\theta\right) = \sigma\varphi\theta \qquad \sigma\varphi\left(\frac{\pi}{2}-\theta\right) = \varepsilon\varphi\theta$$

Remark: 1. In all transformations based on the previous general rules, we assume that the angle θ lies in the first quadrant. However, the results we obtain remain valid in general, no matter how large or small the angle θ is.

2. If the angle is given in degrees, we simply replace π with 180° and $\frac{\pi}{2}$ with 90°, and apply the same rules.

Reminder: In the adjacent diagram you can see the quadrants in which the angles of the forms $k\pi \pm \theta$ and $\frac{k\pi}{2} \pm \theta$ from this section are located, as well as the signs of the trigonometric ratios in each quadrant.



Example 1: Find the trigonometric ratios of the angle 120°.

Solution: Since 120°=180°−60°, we have:

$ημ120^0=ημ(180^o-60^o)=ημ60^o=\frac{\sqrt{3}}{2}$. $σuν120^0=σuν(180^o-60^o)=-σuν60^o=-\frac{1}{2}$

$εφ120^0=εφ(180^o-60^o)=-εφ60^o=-\sqrt{3}$. $σφ120^0=σφ(180^o-60^o)=-σφ60^o=-\frac{\sqrt{3}}{3}$.

Second method: We could also write 120°=90°+30° and then:

$ημ120^0=ημ(90^o+30^o)=σuν30^o=\frac{\sqrt{3}}{2}$. $σuν120^0=σuν(90^o+30^o)=-ημ30^o=-\frac{1}{2}$, etc.

Since we start from 90°, that is $\frac{\pi}{2}$ rad, the type of trigonometric ratio changes.

In both methods, the angle lies in the second quadrant, where ημ is positive and the other trigonometric ratios are negative.

Example 2: Find the trigonometric ratios of the angle 1305°.

Solution: First divide 1305 by 360:

1305	360
225	6

Therefore: 1305°=6·360°+225°, so the trigonometric ratios of 1305° are the same as those of 225°. Since 225°=180°+45°, the angle lies in the third quadrant. Therefore:

$ημ225^0=ημ(180^o+45^o)=-ημ45^o=-\frac{\sqrt{2}}{2}$. $συν225^0=συν(180^o+45^o)=-συν45^o=-\frac{\sqrt{2}}{2}$

$εφ225^0=εφ(180^o+45^o)=εφ45^o=1$ $σφ225^0=σφ(180^o+45^o)=σφ45^o=1$.

Example 3: Find the trigonometric ratios of $\frac{5π}{3}$.

Solution: We note that: $\frac{5π}{3}=2π-\frac{π}{3}$, so the angle lies in the fourth quadrant, where ...... is positive and the remaining trigonometric ratios are negative. Therefore, as in Example 1:

$ημ\frac{5π}{3}=ημ(2π-\frac{π}{3})=$................................ $συν\frac{5π}{3}=συν(2π-\frac{π}{3})=$..............................

$εφ\frac{5π}{3}=εφ(2π-\frac{π}{3})=$................................ $σφ\frac{5π}{3}=σφ(2π-\frac{π}{3})=$.................................

Example 4: Find the trigonometric ratios of $-\frac{103π}{6}$.

Solution: First, divide 103 by 6:

$$\begin{array}{r|l} 103 & 6 \\ \hline 1 & 17 \end{array}$$

Therefore: $-\frac{103π}{6}=-\frac{(6\cdot 17+1)π}{6}=-\frac{6\cdot 17π+π}{6}=-17π-\frac{π}{6}=-18π+π-\frac{π}{6}$, so the trigonometric ratios of $-\frac{103π}{6}$ are the same as those of $π-\frac{π}{6}$. The angle $π-\frac{π}{6}$ lies in the second quadrant, so:

$ημ(-\frac{103π}{6})=ημ(π-\frac{π}{6})=$................................ $συν(-\frac{103π}{6})=συν(π-\frac{π}{6})=$..............................

$εφ(-\frac{103π}{6})=εφ(π-\frac{π}{6})=$................................ $σφ(-\frac{103π}{6})=σφ(π-\frac{π}{6})=$.................................

Example 5: Prove that: $\frac{ημ(5π+ω)\cdot συν(7π-ω)\cdot ημ(\frac{5π}{2}-ω)\cdot συν(\frac{7π}{2}+ω)}{σφ(5π+ω)\cdot ημ(7π-ω)\cdot συν(\frac{5π}{2}-ω)\cdot σφ(\frac{7π}{2}+ω)}=ημ^2ω-1.$

Solution: We compute the trigonometric ratios in the left-hand side:
$ημ(5π+ω)=ημ(4π+π+ω)=ημ(π+ω)=-ημω$, $σφ(5π+ω)=σφ(4π+π+ω)=σφ(π+ω)=σφω$,
$συν(7π-ω)=συν(6π+π-ω)=συν(π-ω)=-συνω$, $ημ(7π-ω)=ημ(6π+π-ω)=ημ(π-ω)=ημω$,
$ημ(\frac{5π}{2}-ω)=ημ(2π+\frac{π}{2}-ω)=ημ(\frac{π}{2}-ω)=συνω$, $συν(\frac{5π}{2}-ω)=συν(2π+\frac{π}{2}-ω)=συν(\frac{π}{2}-ω)=ημω$,
$συν(\frac{7π}{2}+ω)=συν(2π+\frac{3π}{2}+ω)=συν(\frac{3π}{2}+ω)=ημω$, $σφ(\frac{7π}{2}+ω)=σφ(2π+\frac{3π}{2}+ω)=σφ(\frac{3π}{2}+ω)=-εφω$,
therefore the requested expression can be written equivalently as:

$$\frac{-\cancel{ημω}\cdot(-συνω)\cdot συνω\cdot\cancel{ημω}}{σφω\cdot\cancel{ημω}\cdot\cancel{ημω}\cdot(-εφω)}=ημ^2ω-1 \Leftrightarrow \frac{συν^2ω}{-σφω\cdot εφω}=ημ^2ω-1 \Leftrightarrow$$
$$\Leftrightarrow -συν^2ω=ημ^2ω-1 \Leftrightarrow 1=ημ^2ω+συν^2ω$$

which holds by the basic trigonometric identities. We also used that $σφω\cdot εφω=1$.

Exercises

1. Find the trigonometric ratios of the following angles:

i) 120^o ii) 495^o iii) 1200^o iv) -2850^o v) $\frac{5π}{6}$ rad vi) $\frac{11π}{3}$ rad

vii) $-\frac{\pi}{4}$ rad   viii) $\frac{187\pi}{6}$ rad   ix) $\frac{21\pi}{4}$ rad   x) $-\frac{35\pi}{3}$ rad

2. Simplify the expression: $\frac{\sigma\upsilon\nu(-\alpha)\cdot\sigma\upsilon\nu(180^o+\alpha)}{\eta\mu(-\alpha)\cdot\eta\mu(90^o+\alpha)}$.
3. Simplify the expression: $\frac{\varepsilon\varphi(\pi-x)\cdot\sigma\upsilon\nu(2\pi+x)\cdot\sigma\upsilon\nu(\frac{9\pi}{2}+x)}{\eta\mu(13\pi+x)\cdot\sigma\upsilon\nu(-x)\cdot\sigma\varphi(\frac{21\pi}{2}-x)}$.
4. Prove that the following expression has a constant value and find that value:

$$\eta\mu^2(\pi-x)+\sigma\upsilon\nu(\pi-x)\cdot\ \sigma\upsilon\nu(2\pi-x)+2\eta\mu^2(\frac{\pi}{2}-x).$$

5. Using the results from Exercise 1, compute the value of the following expression:
$\frac{\eta\mu 495^o\cdot\sigma\upsilon\nu 120^o+\sigma\upsilon\nu 495^o\cdot\sigma\upsilon\nu(-120^o)}{\varepsilon\varphi(-120^o)+\varepsilon\varphi 495^o}$.
6. **True / False.** For each of the following state whether it is True (Σ) or False (Λ).
   i) $\eta\mu^2 20^o+\eta\mu^2 70^o=1$.
   ii) $\eta\mu(x-\pi)=-\eta\mu x$.
   iii) For every angle x it holds that $\sigma\upsilon\nu(\frac{\pi}{6}-x)-\eta\mu(\frac{\pi}{3}+x)=0$.
7. Match each trigonometric ratio in the 1st column with the equal value in the 2nd column.

1st column	2nd column
1. $\eta\mu 120^o$	a. $-\sqrt{3}$
2. $\sigma\upsilon\nu 150^o$	b. $-\frac{\sqrt{3}}{2}$
3. $\eta\mu 210^o$	c. $-\frac{\sqrt{3}}{3}$
4. $\sigma\upsilon\nu 300^o$	d. $-\frac{1}{2}$
5. $\varepsilon\phi 210^o$	e. $\frac{1}{2}$
6. $\sigma\phi 300^o$	f. $\frac{\sqrt{3}}{3}$
7. $\varepsilon\phi 300^o$	g. $\frac{\sqrt{3}}{2}$
8. $\sigma\phi 210^o$	h. $\sqrt{3}$