3.4 Trigonometric Functions

1. Periodic Functions

A **periodic function** is a function y=f(x) whose values repeat whenever x increases or decreases by a fixed positive number T>0. Many functions describing the evolution of a quantity over time are periodic, as we can see in the following examples.

Example 1: The ship route

A ship repeatedly travels between ports A and B.

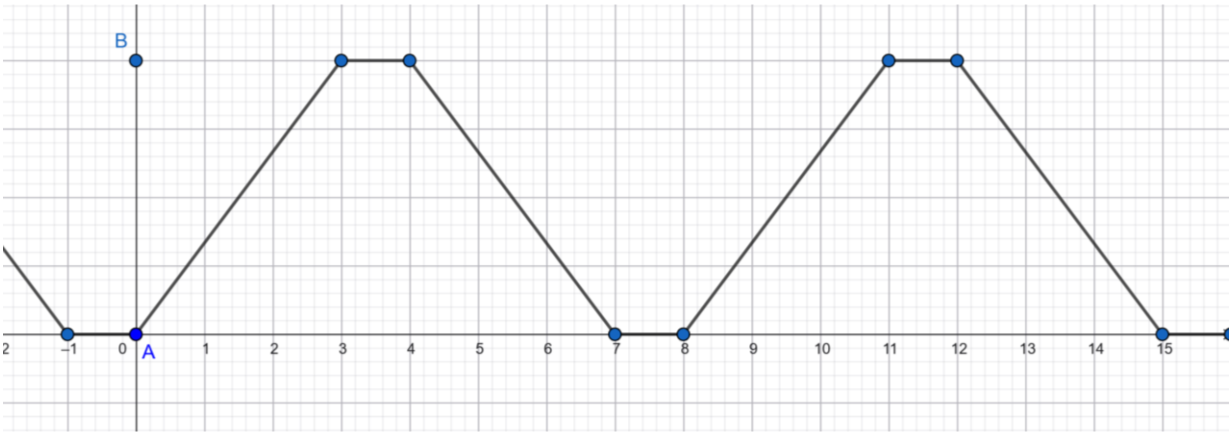
It departs from **A at 12:00**, moves toward **B** with constant speed, and needs **3 hours** to arrive.

Once it reaches **B**, it stops for **1 hour** and then returns to **A**, again in **3 hours**, where it makes another **1-hour stop**.

After that, it repeats exactly the same process.

If we measure time t in hours starting from the ship's departure at 12:00, then the function f(t) , which gives the distance of the ship from A, shows a clearly **repeating** pattern of change: it increases linearly for 3 hours, remains constant for 1 hour, decreases linearly for 3 hours, remains constant again for 1 hour… and then this cycle begins again.

This repeating pattern makes the function periodic. Its graph is shown in the figure below.



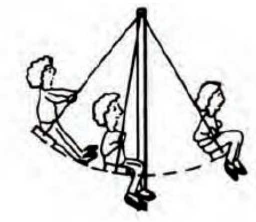
Example 2: The motion of a child on a swing

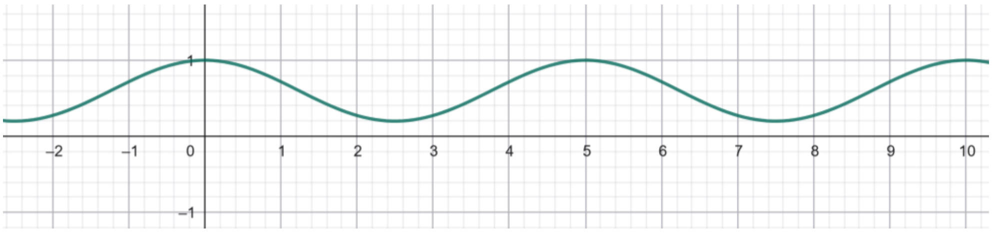
A child sits on a swing and moves back and forth.

If we study the distance of the child's feet from the ground as a function of time, we see that the height goes up and down in a pattern that **constantly repeats**.

The function h(t), which shows the distance of the child's feet from the ground, displays successive maximum and minimum values at regular time intervals, following the natural motion pattern of the swing.

Although the exact form of h(t) may be quite complex, its basic characteristic — the constant repetition of the same pattern with equal time duration — remains clear. This is also visible in the graph of the function h(t) in the figure below.





In the two examples above, we observe a common property:

- The variation of the function **repeats again and again**.
- The time interval in which this repetition occurs is **constant**.
- If we shift the graph horizontally by this specific time interval, the new graph coincides with the original one.

This leads us to the classical definition:

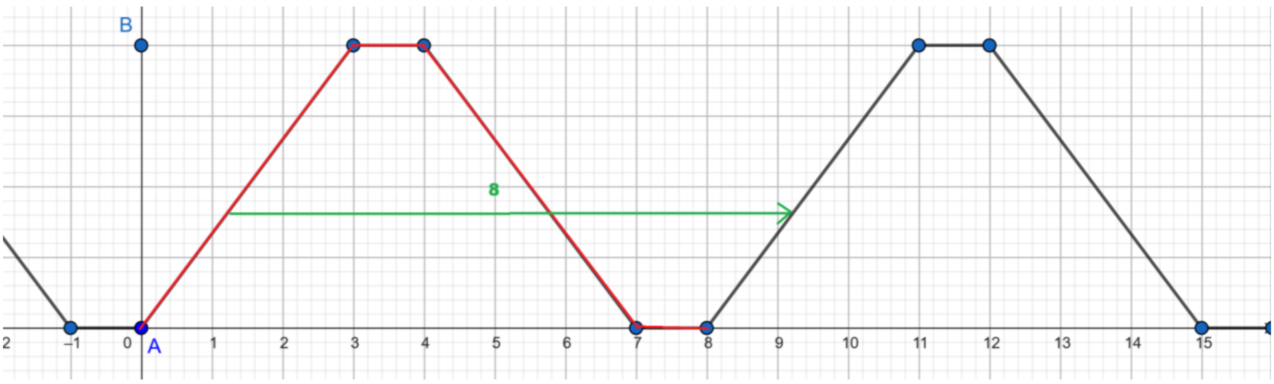
Definition

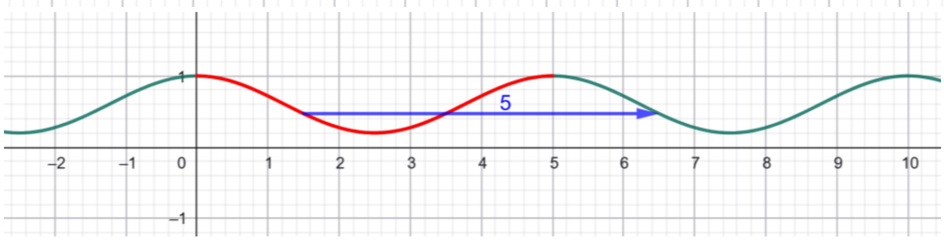
A function f:A→IR is called **periodic** if there exists a real number T>0 such that, for every x in the domain, the following hold:

- The numbers x+T and x−T also belong to the domain, and
- f(x+T)=f(x−T)=f(x).

The number T is called the **period** of the function.

We observe that if a function is periodic with period T>0, then its graph is composed of identical segments, each of width T, which repeat continuously both to the right and to the left. From the diagrams below, we identify that the period of the function describing the motion of the ship in Example 1 is T = ... hours, while the period of the function h(t) in Example 2 is T = ... seconds.





2. The basic trigonometric functions

The trigonometric functions are initially defined for angles (in rad), but they are also used very frequently in other applications, such as oscillations or alternating current in Physics, as functions of a real variable. For this reason, to each real number x we assign the value of each trigonometric ratio of the angle x rad, and we simply write ημx, συνx, εφx, σφx. Specifically:

The function $f(x)=\etaμx$ assigns to every real number x the sine of the angle x rad. From formulas **(3)** of section 3.1, it follows that $f(x+2\pi)=f(x-2\pi)=f(x)$, therefore f is periodic with period $T=2\pi$.

The function $g(x)=συνx$ is defined in the same way for every real number x. From formulas **(3)** of section 3.1, it follows that $g(x+2\pi)=g(x-2\pi)=g(x)$, therefore g is also periodic with period $T=2\pi$.

The function $h(x)=εφx$. From **Question 2** of the activity **"Axis of Tangents"** in section 3.1 (https://tinyurl.com/bdh6u4fh), we can determine that the tangent is defined for every $x\neq k\pi+\frac{\pi}{2}$ where k is an integer, that is, whenever $συνx\neq 0$. From formulas **(3)** of section 3.1, it also follows that $h(x+2\pi)=h(x-2\pi)=h(x)$. However, we have already seen something stronger: in section 3.3, Example 1.III, we saw that $εφ(\pi+x)=εφx$, and therefore $h(x+\pi)=h(x-\pi)=h(x)$. Thus, h is periodic with period $T=\pi$.

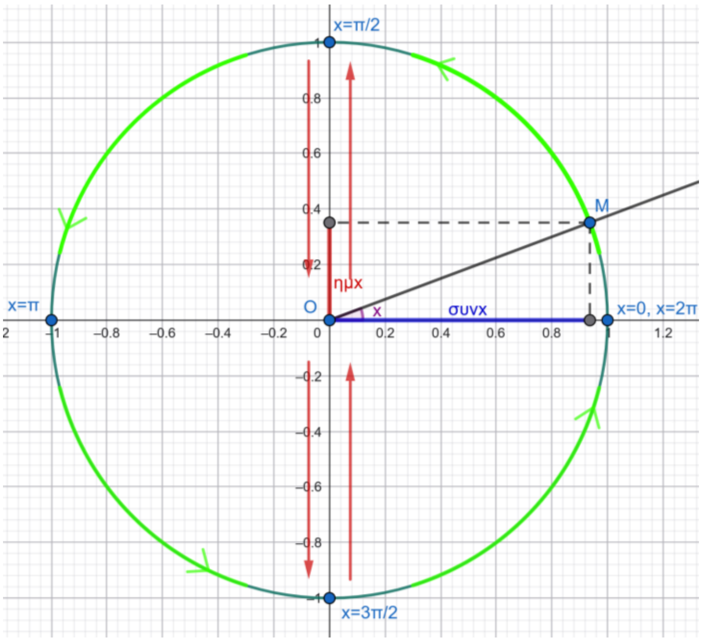
With similar reasoning, we can see that the function $\phi(x)=σφx$ is defined for every real number x for which $ημx\neq 0$, that is, for every $x\neq k\pi$ where k is an integer. It also holds that $σφ(\pi+x)=σφx$, so this function is also periodic with period $T=\pi$.

Next, we will examine in detail the basic properties of the functions ημx, συνx, εφx, and σφx.

3. The function f(x)=ημx

- It has domain the set $\mathbb{R}$ of real numbers.
- Its "range" is the interval $[-1,1]$. In practice, this means that $-1\leq f(x)\leq 1$ for every $x\in\mathbb{R}$, as we have seen in section 3.1.
- It is periodic with period 2π. Therefore, we can study it on any interval of length one period, for example on $[0,2\pi]$.

- As we observe on the trigonometric circle, when we increase the angle x from 0 to $\frac{\pi}{2}$, that is, in the 1st quadrant, ημx also increases. In other words, f is strictly increasing on $[0, \frac{\pi}{2}]$. Continuing to increase x and moving to the remaining quadrants, we observe that:
  In the 2nd quadrant, ημx decreases, meaning that f is strictly decreasing on $[\frac{\pi}{2},\pi]$.
  In the 3rd quadrant, ημx continues to decrease, so f is strictly decreasing on $[\pi, \frac{3\pi}{2}]$.

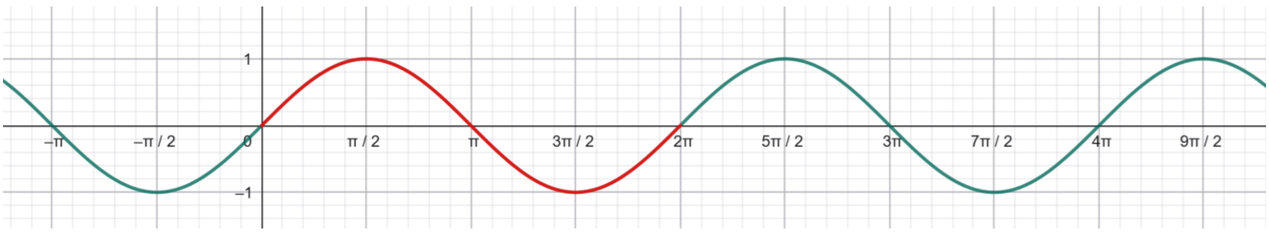


  In the 4th quadrant, ημx increases again, so f is strictly increasing on $[\frac{3\pi}{2},2\pi]$.
  You can confirm these conclusions by using the activity **Monotonicity of ημx and συνx on the trigonometric circle**, which you can find here: https://tinyurl.com/46yzduf6.
- The function f(x)=ημx becomes equal to 0 at x=0, x=π, and x=2π. It attains a maximum at $x=\frac{\pi}{2}$, where $f(\frac{\pi}{2})=1$, and a minimum at $x=\frac{3\pi}{2}$, where $f(\frac{3\pi}{2})=-1$.
  These conclusions about monotonicity and extrema are summarized in the table below:

x	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
ημx		↗		↘		↘		↗	
			Maximum value 1				Minimum value −1		

- It is an odd function, since for every x∈IR, we have f(−x)=ημ(−x)=−ημx=−f(x).

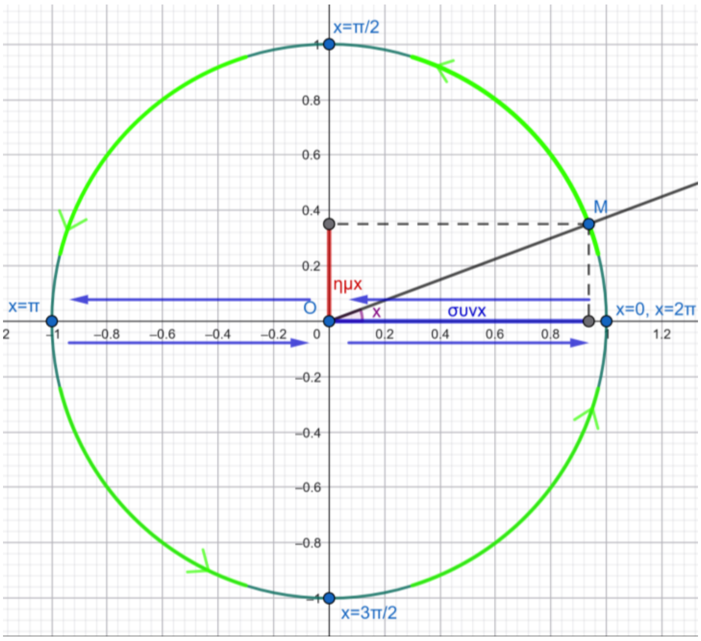
Taking all of the above into account, the graph of f(x)=ημx is the following curve, known as the sine curve:



4. The function g(x)=συνx

- It has domain the set IR of real numbers.
- Its "range" is the interval [−1,1], since here as well we have −1≤g(x)≤1 for every x.

- It is periodic with period 2π. Therefore, we can study it on any interval of length one period, for example on [0,2π].
- As we observe on the trigonometric circle, when we increase the angle x from 0 to $\frac{\pi}{2}$, that is, in the 1st quadrant, συνx decreases. In other words, g is strictly decreasing on $[0, \frac{\pi}{2}]$. Continuing to increase x and moving to the remaining quadrants, we observe that:
  In the 2nd quadrant, συνx continues to decrease, meaning that g is strictly decreasing on $[\frac{\pi}{2},\pi]$.
  In the 3rd quadrant, συνx remains negative, but it begins to increase; therefore, g is strictly increasing on $[\pi, \frac{3\pi}{2}]$.
  In the 4th quadrant, συνx continues to increase, so g is strictly increasing on $[\frac{3\pi}{2},2\pi]$.
  You can also confirm these conclusions using the activity **Monotonicity of ημx and συνx on the trigonometric circle**, as before.



- The function g(x)=συνx becomes equal to 0 at $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$. It attains a maximum at x=0 and at x=2π, with g(0)=g(2π)=1, and a minimum at x=π, with g(π)=−1.
  These conclusions about monotonicity and extrema are summarized in the table below:

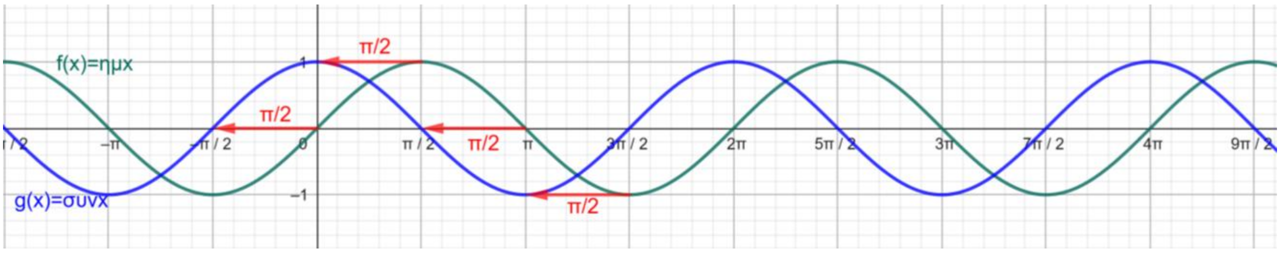
x	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
συνx		↘		↘		↗		↗	
	Maximum value 1				Minimum value −1				Maximum value 1

- It is an even function, since for every x∈IR we have g(−x)=συν(−x)=συνx=g(x).

Taking all of the above into account, the graph of g(x)=συνx is the following:

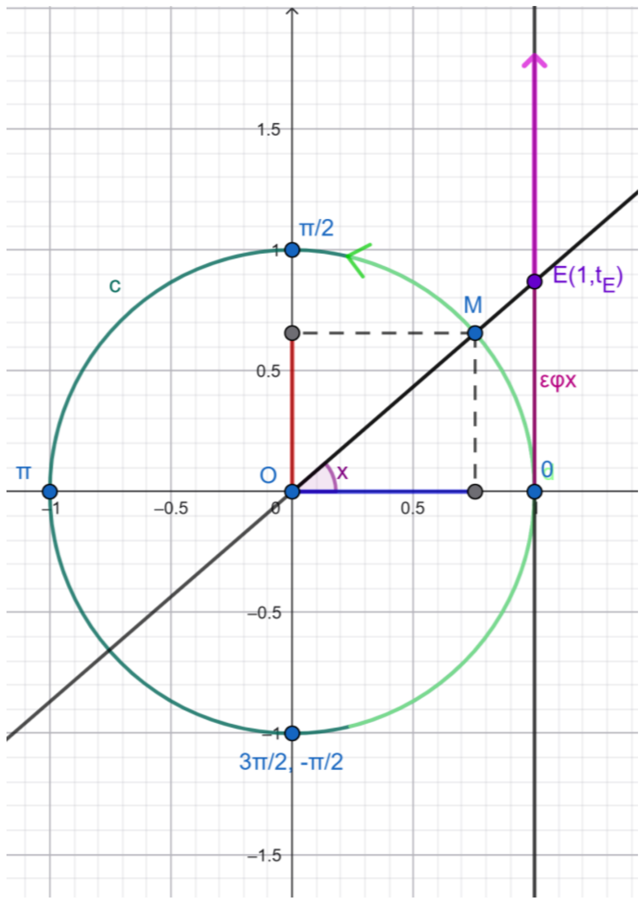
Remark

If we compare the graphs of $f(x)=ημx$ and $g(x)=συνx$ on the same figure, we can see that the graph of g is obtained by shifting the graph of f by $\frac{π}{2}$ units to the left.



This happens because $f(x+\frac{π}{2})=ημ(\frac{π}{2}+x)=συνx=g(x)$, as follows from the reduction formulas to the 1st quadrant, in combination with what we studied in section **2.2**.

5. The function h(x)=εφx



- As we have seen previously, εφx is defined for every $x \neq kπ+\frac{π}{2}$ where k is an integer. Therefore, the function h has as its domain the set $\mathbb{R}_1=\mathbb{R}-\{kπ+\frac{π}{2} \mid k \in \mathbb{Z}\}$ of real numbers for which $συνx \neq 0$.
- It is periodic with period π. Therefore, we can study it on an interval of length π, for example on $(-\frac{π}{2}, \frac{π}{2})$. This interval is open, since h is not defined at its endpoints.
- As we see on the trigonometric circle with the help of the tangent axis, when the angle x increases from $-\frac{π}{2}$ to $\frac{π}{2}$, the point E moves upwards; therefore, εφx increases. Moreover, for any position of the point E on the tangent axis, the line OE intersects the trigonometric circle at a point M, so there exists an angle $x \in (-\frac{π}{2}, \frac{π}{2})$ such that $εφx=t_E$. Thus, h is strictly increasing on $(-\frac{π}{2}, \frac{π}{2})$ and its range is the entire set $\mathbb{R}$.
- The function $h(x)=εφx$ is equal to 0 for x=0. It also increases without bound, that is, it "tends to +∞" as x approaches $\frac{π}{2}$ from smaller values, and correspondingly it "tends to −∞" as x approaches $-\frac{π}{2}$ from larger values.
- It is an odd function, since for every $x \in \mathbb{R}_1$, we have $h(-x)=εφ(-x)=-εφx=-h(x)$.

Taking the above into account, the graph of h(x)=εφx is as follows:

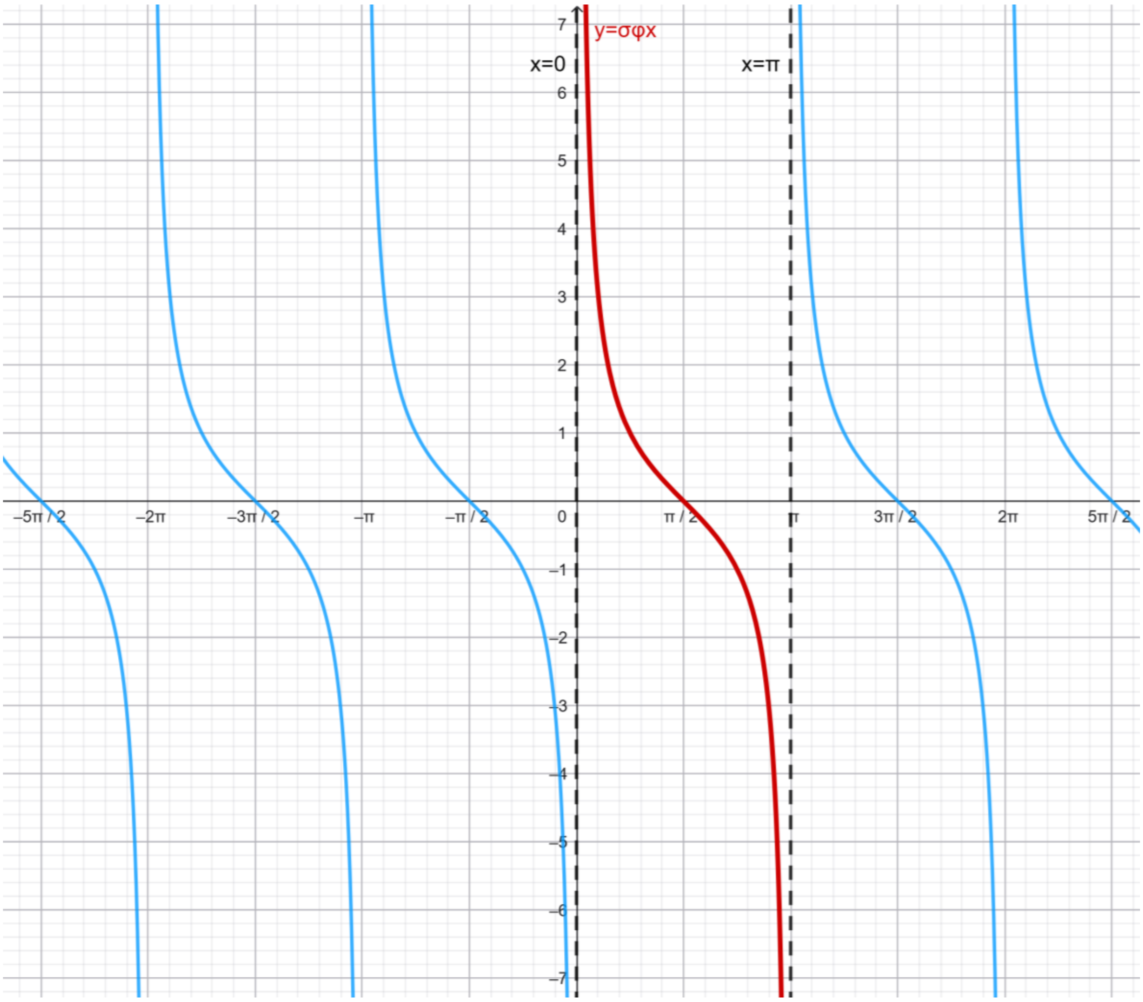
The lines $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$, as well as any line of the form $x=k\pi+\frac{\pi}{2}$ for $k\in\mathbb{Z}$, are called vertical asymptotes of the graph of h.

6. The function ϕ(x)=σϕx

In a similar manner, we can conclude the following results for the function ϕ(x)=σϕx:

- It has a domain of definition given by the set $\mathbb{R}_2=\mathbb{R}-\{k\pi \mid k\in\mathbb{Z}\}$ of real numbers for which ημx≠0 and its "range" is the entire $\mathbb{R}$.
- It is periodic with period π, therefore we can study it on an interval of length π, e.g. on (0, π). The interval is open because the function is not defined at its endpoints.
- It is an odd function, since for every $x\in\mathbb{R}_2$, ϕ(−x)=σϕ(−x)=−σϕx=−ϕ(x).

The graph of ϕ(x) = σϕx is as follows:



From the graph, we can deduce the following conclusions:

- ϕ is strictly ____________ on (0,π), as well as on every other interval of its domain.
- It is equal to 0 for x = ___. It "tends to ___" as x approaches 0 from higher values, and "tends to ___" as x approaches π from lower values.
- The straight lines x = ___, x = ___, as well as any other line of the form x = kπ for $k \in \mathbb{Z}$, are vertical asymptotes of the graph of ϕ.

[The completion of the blanks is left as an exercise.]

7. Functions of the form f(x)=ρημ(ωx) and g(x)=ρσυν(ωx)

A function of the form f(x)=ρημ(ωx), with ρ>0 and ω>0, also has a graph similar to the sine curve, which passes through the point O(0,0). The parameters ρ and ω affect the maximum and minimum values and the period of the function as follows:

- The maximum value of f is equal to ρ and the minimum value is equal to −ρ.
- The period is equal to $T=\frac{2\pi}{\omega}$.

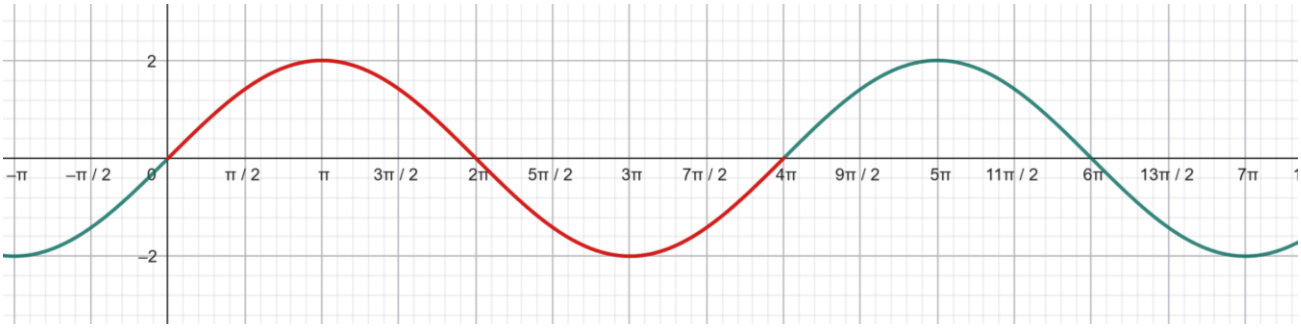
The same conclusions apply to the maximum value and the period of a function of the form g(x)=ρσυν(ωx), when ρ>0 and ω>0, with the difference that for x=0 the function attains its maximum value.

If ρ<0 (while ω remains positive), the following changes occur:
On the one hand, the maximum value of f and g becomes |ρ| and the minimum value becomes −|ρ|.
On the other hand, the function f(x)=ρημ(ωx) is strictly decreasing on the first interval to the right of the point O(0,0), while the function g(x)=ρσυν(ωx) attains its minimum value at x=0.

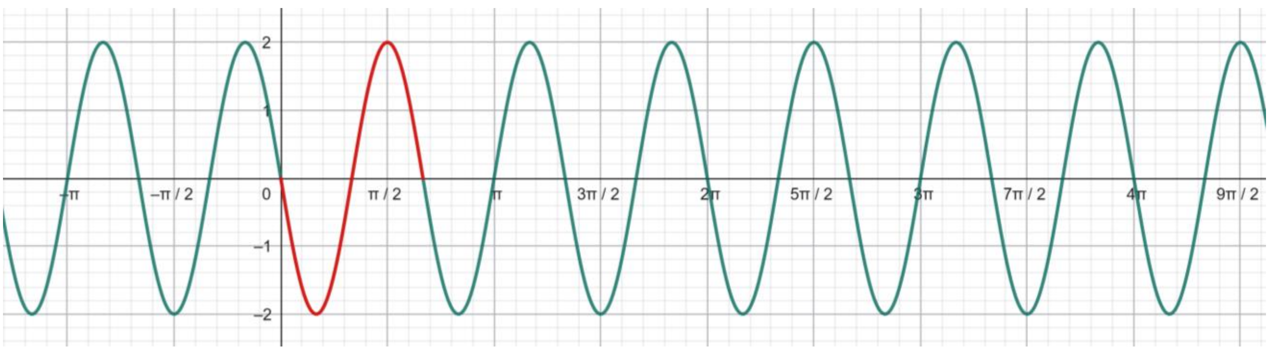
You can verify the above conclusions for functions of these forms using the activities **y=ρημ(ωx)** and **y=ρσυν(ωx)**, which you can find here: https://tinyurl.com/46yzduf6.

Example 1: Given the function f(x)=2ημ$\frac{x}{2}$. Find its maximum and minimum value, its period, and draw its graph.

Solution: The function is of the form f(x)=ρημ(ωx) with ρ=2 and ω=$\frac{1}{2}$. Therefore, it has maximum value 2, minimum value −2, and period T=$\frac{2\pi}{\frac{1}{2}}$=4π. Its graph is shown below:



Example 2: Find the function whose graph is shown below.

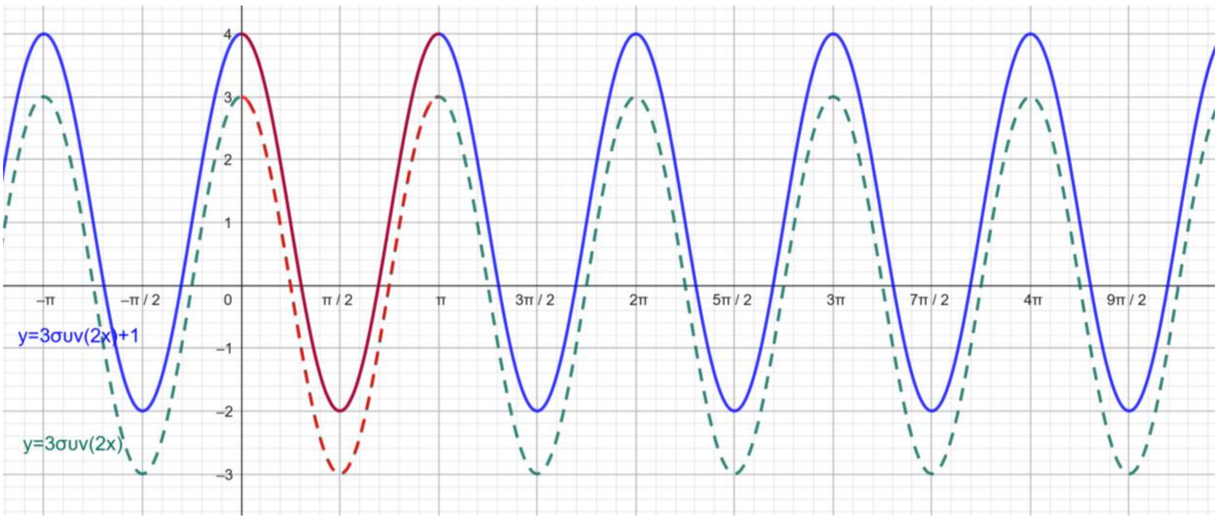


Solution: From the graph we see that the maximum value is 2 and the minimum value is −2, while the interval [0,2π] contains 3 periods of the function. Therefore, T=$\frac{2\pi}{3}$. Moreover, the curve passes through the point O(0,0) and, on the first interval of monotonicity to the right of

$O(0,0)$, it is strictly decreasing. Thus, the function is of the form $f(x)=\rho ημ(\omega x)$ with $\rho<0$. Hence, $|\rho|=2$, so $\rho=-2$ and $\frac{2\pi}{\omega}=\frac{2\pi}{3}$, so $\omega=3$. Therefore $f(x)=-2ημ(3x)$.

Example 3: Find the maximum value, the minimum value, and the period of the function $f(x)=3συν(2x)+1$, and draw its graph.

Solution: We start with the function $g(x)=3συν(2x)$. This function has maximum value 3, minimum value -3, and period $T=\frac{2\pi}{2}=\pi$. The function $f(x)=g(x)+1$ therefore has maximum value $3+1=4$ and minimum value $-3+1=-2$. Its graph is obtained by shifting the graph of g upward by 1 unit, so its period is also $T=\pi$. Combining the above, the graph of f is shown below.

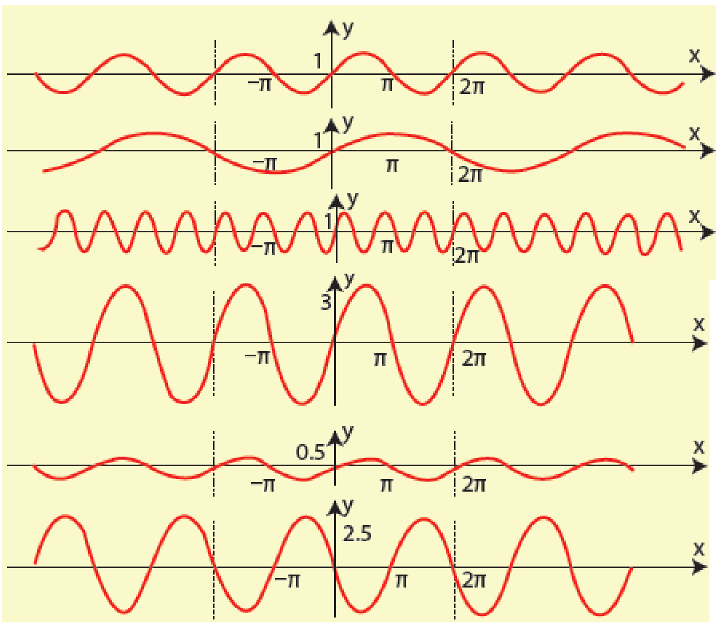


Exercises

1. Find the maximum value, the minimum value, and the period of the following functions:

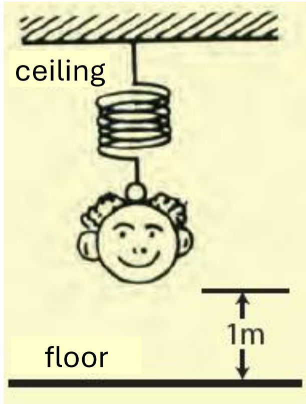
   i) $f(x)=2ημx$ ii) $g(x)=2ημx-1$ iii) $h(x)=2συν\frac{x}{2}$ iv) $\phi(x)=2συν\frac{x}{2}+1$

2. Find the equation of each of the following sine-type curves:

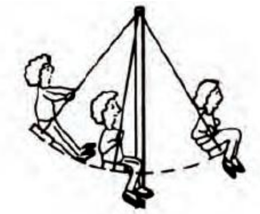


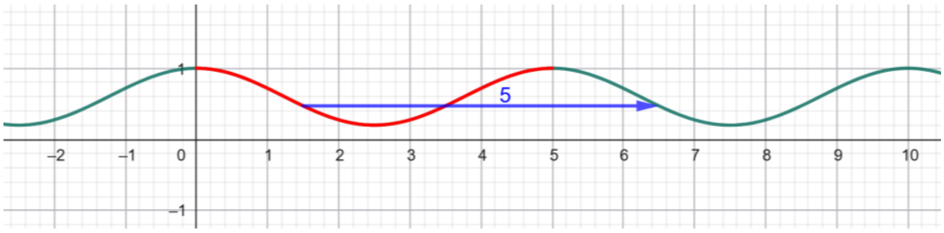
3. The tide in a coastal area is described by the function $y=3\cdot \eta\mu(\frac{\pi}{6}\cdot t)$, where y is the height of the water level in meters and t is the time in hours.
   i) Find the difference in height between the highest tide and the lowest ebb.
   ii) Find the period of the function and draw its graph for $0 \leq t \leq 12$.

4. A toy is hanging from the ceiling by a spring and is at a distance of 1 m from the floor. When it is pulled downward, it starts moving up and down, and its height from the floor (in meters) is given by $h=1-\frac{1}{3}\sigma\upsilon\nu(3t)$, where t is the time in seconds.
   i) Find the difference between the maximum and the minimum height.
   ii) Find the period of the oscillation.
   iii) Draw the graph of the function for $0\leq t\leq 2\pi$.



5. Consider the function $g(t)=\alpha+\rho\sigma\upsilon\nu(\omega t)$, where $\rho>0$ and $\omega>0$.
   i) Prove that its maximum value is $\alpha+\rho$ and its minimum value is $\alpha-\rho$, and find its period T.
   ii) In Example 2 of Paragraph 1 of this section, we saw that the function describing the distance from the ground of a child’s feet while swinging has the following graph, with period T=5 sec:





Given that the maximum distance of the child’s feet from the ground is 1 meter and the minimum distance is 0.2 meters, find the formula of the function g(t) that describes this motion.