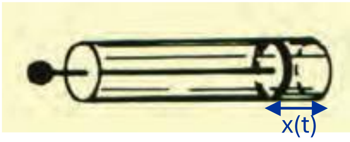
3.5 Trigonometric equations

In many cases, the study of problems that refer to phenomena described by trigonometric functions leads to the formulation of equations in which the unknown appears inside a trigonometric number.

Example: (From **Exercise B4** of the Algebra textbook, **p. 83)**).
The distance x(t) of a piston that moves back and forth inside a cylinder from one end of the cylinder, measured in meters, is given by the function x(t) = 0,1 + 0,1·ημ(3t), where t denotes time in seconds. At which time instants is this distance equal to 0,15 meters?



If we attempt to answer this question, we must solve the equation x(t) = 0,15 with respect to t. Substituting, we obtain successively:

$$x(t) = 0{,}15 \Leftrightarrow 0{,}1 + 0{,}1 \cdot \eta\mu(3t) = 0{,}15 \Leftrightarrow 0{,}1 \cdot \eta\mu(3t) = 0{,}15 - 0{,}1 \Leftrightarrow 0{,}1 \cdot \eta\mu(3t) = 0{,}05$$
$$\Leftrightarrow \eta\mu(3t) = \frac{0{,}05}{0{,}1} \Leftrightarrow \eta\mu(3t) = \frac{1}{2}$$

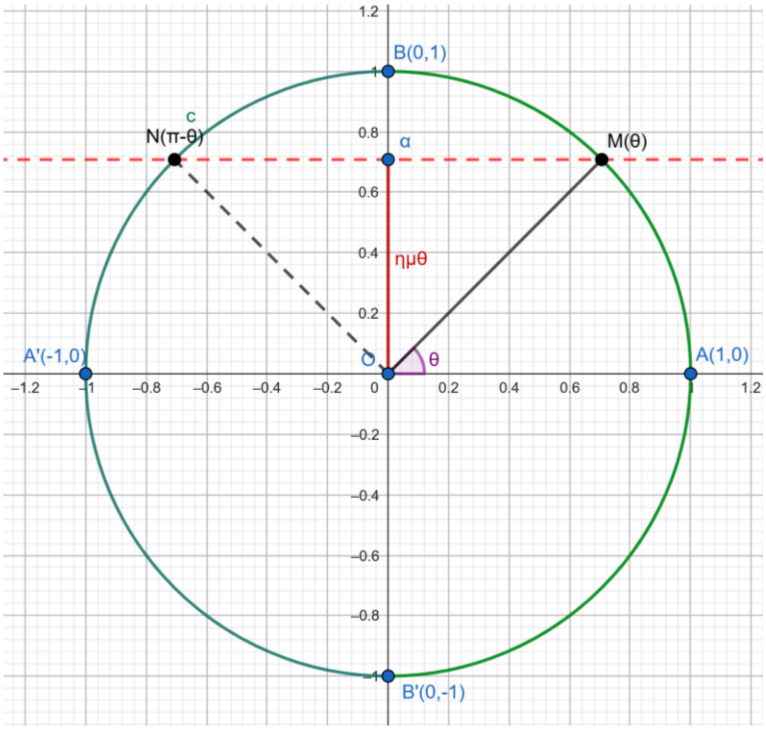
This equation is not algebraic. The unknown t appears inside a trigonometric number and cannot be solved using the methods available to us so far. In order to find all the time instants that satisfy the condition, we need new tools and a more systematic way of thinking.
This is precisely what leads us to the study of the basic **trigonometric equations**, which is the subject of this section.

1. The equation ημx = α

We will study the equation ημx = α with the help of the trigonometric circle.
We recall that, for every angle θ, the value ημθ represents the y-coordinate of the point M on the trigonometric circle that corresponds to the angle θ.
Therefore, the equation ημx = α is equivalent to the following geometric problem:
to find the points of the trigonometric circle whose y-coordinate is equal to α.

We observe that, since for every real number x it holds that $-1 \le \eta\mu x \le 1$, **if α < −1 or α > 1**, then the equation ημx = α has **no solution**.

If $\alpha \in [-1, 1]$, we place the number α on the sine axis and draw a horizontal line, which intersects the trigonometric circle at two points M and N. Let θ be the angle that corresponds to point M. Then $\eta\mu\theta = \alpha$, and all angles of the form $2k\pi + \theta$, where $k \in \mathbb{Z}$, correspond to point M. Moreover, due to the symmetry of the figure with respect to the y-axis, the angle $\pi - \theta$ corresponds to point N, as well as all angles of the form $2k\pi + \pi - \theta$, where $k \in \mathbb{Z}$. Therefore, the equation can be written successively as:



$$\eta\mu x = \alpha \Leftrightarrow \eta\mu x = \eta\mu\theta \Leftrightarrow x = 2k\pi + \theta \text{ or } x = 2k\pi + \pi - \theta \ (k \in \mathbb{Z})$$

Example 1: Solve the equation $\eta\mu x=\frac{1}{2}$.

Solution: From the tables of trigonometric values, we find that $\eta\mu\frac{\pi}{6}=\frac{1}{2}$. Therefore, the equation is written as:

$$\eta\mu x = \frac{1}{2} \Leftrightarrow \eta\mu x = \eta\mu\frac{\pi}{6} \Leftrightarrow x = 2k\pi + \frac{\pi}{6} \text{ or } x = 2k\pi + \pi - \frac{\pi}{6} = 2k\pi + \frac{5\pi}{6} \ (k \in \mathbb{Z})$$

Example 2: Solve the equation $\eta\mu x=-\frac{\sqrt{2}}{2}$.

Solution: We know that $\eta\mu\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ and that opposite angles have opposite sine values. Thus, $\eta\mu(-\frac{\pi}{4})=-\frac{\sqrt{2}}{2}$ and the equation becomes:

$$\eta\mu x = -\frac{\sqrt{2}}{2} \Leftrightarrow \eta\mu x = \eta\mu\left(-\frac{\pi}{4}\right) \Leftrightarrow \begin{matrix} x = 2k\pi + \left(-\frac{\pi}{4}\right) = 2k\pi - \frac{\pi}{4} \text{ or} \\ x = 2k\pi + \pi - \left(-\frac{\pi}{4}\right) = 2k\pi + \pi + \frac{\pi}{4} = 2k\pi + \frac{5\pi}{4} \end{matrix} \ (k \in \mathbb{Z})$$

Example 3: Solve the equation $(6\eta\mu x-3)(2\eta\mu x+4)=0$.

Solution: We have: $6\eta\mu x-3=0$ or $2\eta\mu x+4=0 \Leftrightarrow 6\eta\mu x=3$ or $2\eta\mu x=-4 \Leftrightarrow \eta\mu x=\frac{1}{2}$ or $\eta\mu x=-2$. The second equation is impossible, while the first one is solved as in Example 1. Therefore, the solutions are: $x=2k\pi+\frac{\pi}{6}$ or $x=2k\pi+\frac{5\pi}{6}$ ($k\in \mathbb{Z}$).

Example 4: Solve the equation $\eta\mu(3t) =\frac{1}{2}$. (Continuation of the **Example** from the introduction of this section).

Solution: Now, this equation can be solved as follows:

$$\eta\mu(3t) = \frac{1}{2} \Leftrightarrow \eta\mu(3t) = \eta\mu\frac{\pi}{6} \Leftrightarrow 3t = 2k\pi + \frac{\pi}{6} \text{ or } 3t = 2k\pi + \pi - \frac{\pi}{6} = 2k\pi + \frac{5\pi}{6}$$
$$\Leftrightarrow t = \frac{2k\pi}{3} + \frac{\pi}{18} \text{ or } t = \frac{2k\pi}{3} + \frac{5\pi}{18} \ (k \in \mathbb{Z})$$

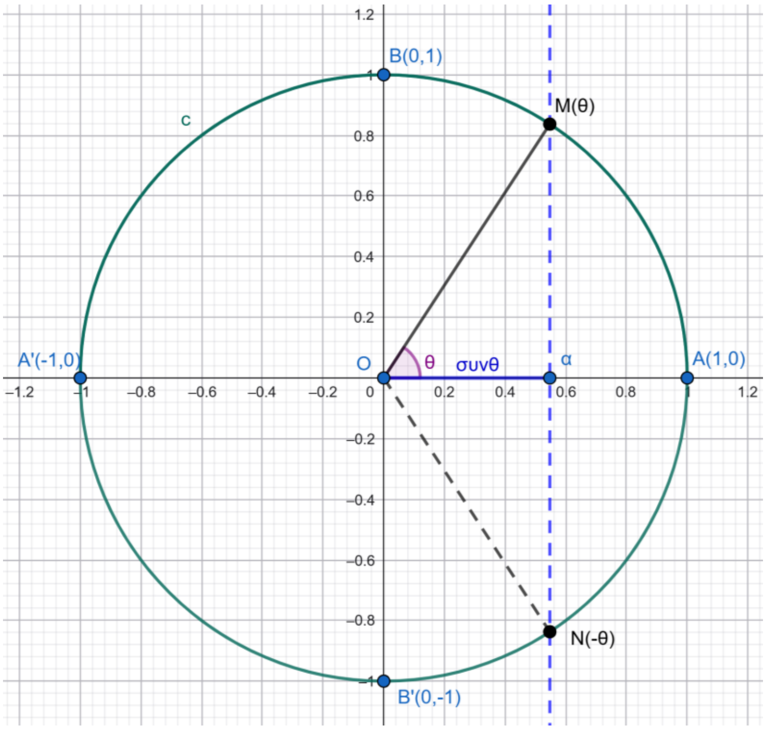
Remark: In some cases, the formulas that give the solutions of a trigonometric equation can be unified. This happens in the equations ημx=1, ημx=−1 and ημx=0, which we present below:

1. The equation **ημx = 1** can be written equivalently as: $\eta\mu x=\eta\mu\frac{\pi}{2} \Leftrightarrow x=2k\pi+\frac{\pi}{2}$ or $x=2k\pi+\pi-\frac{\pi}{2}=2k\pi+\frac{\pi}{2}$ $(k\in \mathbb{Z})$. Therefore, in this case, the formula $x=2k\pi+\frac{\pi}{2}$ $(k\in \mathbb{Z})$ represents all the solutions of the equation.
2. The equation **ημx=−1** can be written equivalently as: $\eta\mu x=\eta\mu\left(-\frac{\pi}{2}\right) \Leftrightarrow x=2k\pi-\frac{\pi}{2}$ or $x=2k\pi+\pi+\frac{\pi}{2}=2k\pi+\frac{3\pi}{2}$ $(k\in \mathbb{Z})$. However, on the trigonometric circle, the two formulas correspond to the same point. Thus, all solutions of the equation are given by either of these formulas, for example by $x=2k\pi-\frac{\pi}{2}$ $(k\in \mathbb{Z})$.
3. The equation **ημx=0** can be written equivalently as: $\eta\mu x=\eta\mu 0 \Leftrightarrow x=2k\pi+0=2k\pi$ or $x=2k\pi+\pi-0=2k\pi+\pi=(2k+1)\pi$ $(k\in \mathbb{Z})$. Thus, the first formula gives all even multiples of π and the second gives all odd multiples of π. Therefore, overall we obtain all integer multiples of π, which are expressed by the unified formula $x=k\pi$ $(k\in \mathbb{Z})$, which gives all the solutions of the equation.

2. The equation συνx = α

We now move on to the equation συνx = α. On the trigonometric circle, for every angle θ, the value συνθ represents the x-coordinate of the point M of the trigonometric circle that corresponds to the angle θ. Therefore, following the same reasoning as before, we must now **find the points of the trigonometric circle whose x-coordinate is equal to α**.
Here as well, for every real number x we have $-1 \leq \sigma\upsilon\nu x \leq 1$. Hence, **if α < −1 or α > 1**, the equation συνx = α has **no solution**.

If α ∈ [−1, 1], we place the number α on the cosine axis and draw a vertical line, which intersects the trigonometric circle at two points M and N. Let θ be the angle corresponding to point M. Then we have συνθ = α, and point M corresponds to all angles of the form 2kπ + θ, where k∈ℤ. Moreover, due to the symmetry of the figure with respect to the x-axis, point N corresponds to the angle −θ, as well as to all angles of the form 2kπ − θ, where k∈ℤ. Consequently, the equation can be written successively as:



$$\text{συν}x = \alpha \Leftrightarrow \text{συν}x = \text{συν}\theta \Leftrightarrow x = 2k\pi + \theta \ \text{ or } \ x = 2k\pi - \theta \ \ (k \in \mathbb{Z})$$

Example 1: Solve the equation συν$x=\frac{\sqrt{3}}{2}$.

Solution: From the tables of trigonometric numbers we find that συν$\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, so the equation is written as:

$$\text{συν}x = \frac{\sqrt{3}}{2} \Leftrightarrow \text{συν}x = \text{συν}\frac{\pi}{6} \Leftrightarrow x = 2k\pi + \frac{\pi}{6} \ \text{ or } \ x = 2k\pi - \frac{\pi}{6} \ (k \in \mathbb{Z})$$

Example 2: Solve the equation συν$x=-\frac{1}{2}$.

Solution: We find that συν$\frac{\pi}{3}=\frac{1}{2}$ and, since supplementary angles have opposite cosine values, we have συν$(\pi-\frac{\pi}{3})=-\frac{1}{2}$, that is συν$\frac{2\pi}{3}=-\frac{1}{2}$. Thus, the equation is written as:

$$\text{συν}x = -\frac{1}{2} \Leftrightarrow \text{συν}x = \text{συν}\frac{2\pi}{3} \Leftrightarrow x = 2k\pi + \frac{2\pi}{3} \ \text{ or } \ x = 2k\pi - \frac{2\pi}{3} \ \ (k \in \mathbb{Z})$$

Example 3: Solve the equation 2συν$^2x-5$συν$x-3=0$.

Solution: We set $y=$συνx and the equation becomes $2y^2-5y-3=0$. We solve the quadratic equation: $\Delta=(-5)^2-4\cdot 2\cdot(-3)=25+24=49$, so $y=\frac{5\pm\sqrt{49}}{2\cdot 2}$, from which we find $y=3$ or $y=-\frac{1}{2}$. Therefore, we obtain the equations συν$x=3$ or συν$x=-\frac{1}{2}$. The first equation has no solution, while the second is solved as in Example 2. Hence, the solutions are: $x=2k\pi+\frac{2\pi}{3}$ or $x=2k\pi-\frac{2\pi}{3}$ ($k\in \mathbb{Z}$).

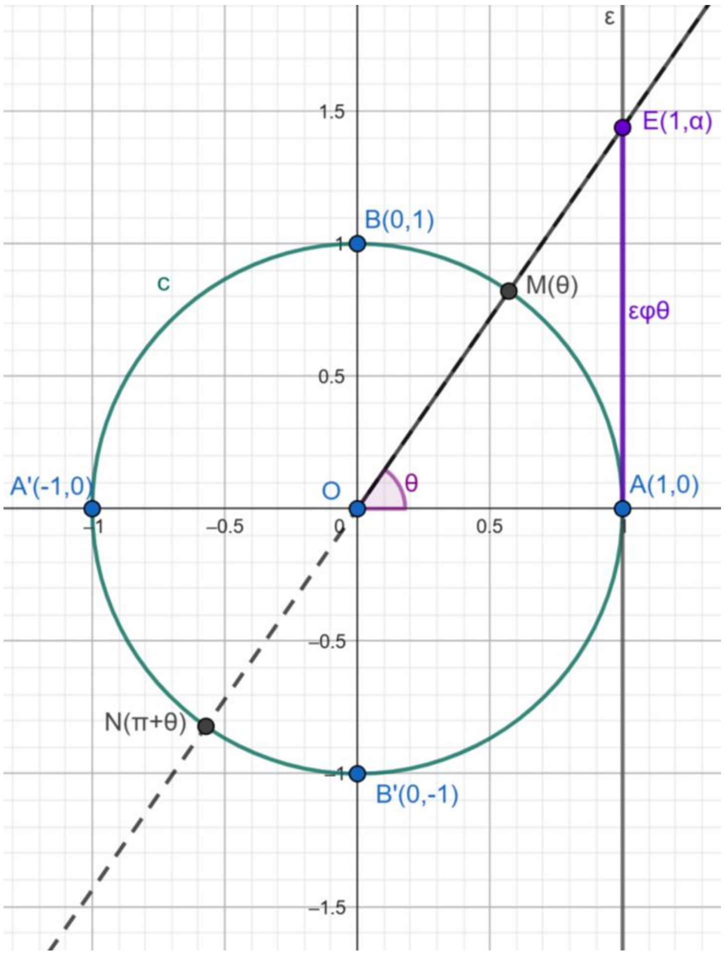
Remark: As in the equation ημx = α, in certain cases the formulas that give the solutions can be unified. This happens in the equations συνx=1, συνx=−1 and συνx=0, which we present below:

1. The equation **συνx=1** can be written equivalently as: συνx=συν0 ⟺ x=2kπ±0=2kπ (k∈ ℤ). Therefore, the formula x=2kπ (k∈ ℤ) gives all the solutions of the equation.
2. The equation **συνx=−1** can be written equivalently as: συνx=συνπ⟺ x=2kπ+π or x=2kπ−π (k∈ ℤ). However, on the trigonometric circle, the two formulas correspond to the same point. Thus, all solutions of the equation are given by either of these formulas, for example by x=2kπ+π (k∈ ℤ).
3. The equation **συνx=0** can be written equivalently as: $συνx=συν\frac{π}{2}$⟺ $x=2kπ+\frac{π}{2}$ or $x=2kπ-\frac{π}{2}=2kπ-π+\frac{π}{2}=(2k-1)π+\frac{π}{2}$ (k∈ ℤ). Thus, the first formula gives the even multiples of π increased by $\frac{π}{2}$ and the second gives the odd multiples of π increased by $\frac{π}{2}$. Therefore, altogether we obtain all integer multiples of π increased by $\frac{π}{2}$, which are expressed by the unified formula $x=kπ+\frac{π}{2}$ (k∈ ℤ), which gives all the solutions of the equation.

3. The equation εφx=α

If α is any real number, we place it on the tangent axis, at the point E(1, α). The ray OE intersects the trigonometric circle at a unique point M, to which there corresponds an angle $θ \in (-\frac{π}{2}, \frac{π}{2})$ such that εφθ = α. The equation can then be written as εφx = εφθ, which has the unique solution x = θ in the interval $(-\frac{π}{2}, \frac{π}{2})$. And since the tangent function is periodic with period π, all solutions of the equation are of the form x=kπ+θ, where k∈ℤ.

(In the figure, the solution x = π + θ is also shown; this is the angle that corresponds to the point N, the second point at which the line OE intersects the trigonometric circle.)



Therefore, the equation has solutions for every α∈IR, which are found as follows:

$$\varepsilon\varphi x = \alpha \Leftrightarrow \varepsilon\varphi x = \varepsilon\varphi\theta \Leftrightarrow x = k\pi + \theta \quad (k \in \mathbb{Z})$$

Example 1: Solve the equation εφx=$\sqrt{3}$.

Solution: From the tables of trigonometric values, we find that εφ$\frac{\pi}{3}$=$\sqrt{3}$. Therefore, the equation can be written as:

$$\varepsilon\varphi x = \sqrt{3} \Leftrightarrow \varepsilon\varphi x = \varepsilon\varphi\frac{\pi}{3} \Leftrightarrow x = k\pi + \frac{\pi}{3} \quad (k \in \mathbb{Z})$$

Example 2: Solve the equation εφx=−1.

Solution: We find that εφ$\frac{\pi}{4}$=1 and, since opposite angles have opposite tangents, we have εφ(−$\frac{\pi}{4}$)=−1. Thus, the equation is written as:

$$\varepsilon\varphi x = -1 \Leftrightarrow \varepsilon\varphi x = \varepsilon\varphi\left(-\frac{\pi}{4}\right) \Leftrightarrow x = k\pi - \frac{\pi}{4} \quad (k \in \mathbb{Z})$$

Example 3: Solve the equation εφ(2x−$\frac{\pi}{3}$)=1.

Solution: We find that εφ$\frac{\pi}{4}$=1, so the equation becomes:

$$\varepsilon\varphi\left(2x - \frac{\pi}{3}\right) = \varepsilon\varphi\frac{\pi}{4} \Leftrightarrow 2x - \frac{\pi}{3} = k\pi + \frac{\pi}{4} \Leftrightarrow 2x = k\pi + \frac{\pi}{4} + \frac{\pi}{3} \Leftrightarrow 2x = k\pi + \frac{7\pi}{12}$$
$$\Leftrightarrow x = \frac{k\pi}{2} + \frac{7\pi}{24} \quad (k \in \mathbb{Z})$$

4. The equation σφx=α

Using procedures analogous to those applied to the equation εφx = α, we can show that an equation of the form σφx = α has solutions for every α∈IR, given by the same general solution formula:

$$\sigma\varphi x = \alpha \Leftrightarrow \sigma\varphi x = \sigma\varphi\theta \Leftrightarrow x = k\pi + \theta \quad (k \in \mathbb{Z})$$

Example: Solve the equation σφx=−$\sqrt{3}$.

Solution: From the tables of trigonometric values we find that σφ$\frac{\pi}{6}$=$\sqrt{3}$, and, since opposite angles have opposite cotangents, it follows that σφ(−$\frac{\pi}{6}$) =−$\sqrt{3}$. Therefore, the equation can be written as:

$$\sigma\varphi x = -\sqrt{3} \Leftrightarrow \sigma\varphi x = \sigma\varphi\left(-\frac{\pi}{6}\right) \Leftrightarrow x = k\pi - \frac{\pi}{6} \quad (k \in \mathbb{Z})$$

Remark

Since the functions εφx and σφx **are not defined for all real numbers x**, in more complex trigonometric equations – such as those we will study in the next unit – we must check whether the solutions obtained satisfy the corresponding domain restrictions:

- Every angle appearing inside an εφ must be different from kπ+$\frac{\pi}{2}$, where k∈ ℤ, or equivalently, its συν must be different from 0.
- Every angle appearing inside a σφ must be different from kπ, where k∈ ℤ, or equivalently, its ημ must be different from 0.

5. Composite trigonometric equations

In the previous sections, we saw how to solve the basic trigonometric equations ημx=α, συνx=α, εφx=α και σφx=α. In many cases, however, equations arise that cannot be reduced directly to one of the above forms and require additional manipulations. We will deal with such equations in this section.

Depending on the way we approach them, we classify them into the following categories:
I) Searching for solutions within a given interval
II) Use of trigonometric identities
III) Change of trigonometric number (reduction to the first quadrant)

Next, we will see characteristic examples from each category.

I) Equations with solution search in a given interval

A trigonometric equation has infinitely many solutions. However, in many cases we are asked to find only those solutions that belong to a specific interval. In this case, after finding the general form of the solutions, we check which of them belong to the given interval.

Example 1: Solve the equation ημ(3t)=$-\frac{1}{2}$ in the interval [0,2π].
Solution: First, we find the general form of the solutions:

$$\eta\mu(3t) = -\frac{1}{2} \Leftrightarrow \eta\mu(3t) = -\eta\mu\frac{\pi}{6} \Leftrightarrow \eta\mu(3t) = \eta\mu\left(-\frac{\pi}{6}\right) \Leftrightarrow \begin{matrix} 3t = 2k\pi - \frac{\pi}{6} \ \text{ or} \\ 3t = 2k\pi + \pi + \frac{\pi}{6} = 2k\pi + \frac{7\pi}{6} \end{matrix}$$

$$\Leftrightarrow \begin{matrix} t = \frac{2k\pi}{3} - \frac{\pi}{18} \ \text{ or} \\ t = \frac{2k\pi}{3} + \frac{7\pi}{18} \end{matrix} \quad (k \in \mathbb{Z})$$

Next, we check which solutions belong to the interval [0, 2π], that is, which satisfy the inequality 0 ≤ t ≤ 2π. We substitute t from each solution formula and solve the resulting inequalities with respect to k. From the first formula:

$$0 \le \frac{2k\pi}{3} - \frac{\pi}{18} \le 2\pi \Leftrightarrow 0 \le 18\cdot\frac{2k\pi}{3} - 18\cdot\frac{\pi}{18} \le 18\cdot 2\pi \Leftrightarrow 0 \le 12k\pi - \pi \le 36\pi \Leftrightarrow$$
$$\pi \le 12k\pi \le 37\pi \Leftrightarrow \frac{1}{12} \le k \le \frac{37}{12}$$

Since k∈Z, we obtain k=1 or k=2 or k=3. Therefore, the first formula gives the solutions

$$t = \frac{2\pi}{3} - \frac{\pi}{18} = \frac{11\pi}{18}, \qquad t = \frac{4\pi}{3} - \frac{\pi}{18} = \frac{23\pi}{18}, \qquad t = 2\pi - \frac{\pi}{18} = \frac{35\pi}{18}$$

all of which belong to the interval [0,2π]. Similarly, from the second formula:

$$0 \leq \frac{2k\pi}{3} + \frac{7\pi}{18} \leq 2\pi \Leftrightarrow 0 \leq 12k\pi + 7\pi \leq 36\pi \Leftrightarrow -7\pi \leq 12k\pi \leq 29\pi \Leftrightarrow -\frac{7}{12} \leq k \leq \frac{29}{12}$$

Since k∈Z, we obtain k=0 or k=1 or k=2. Thus, the second formula gives the solutions

$$t = \frac{7\pi}{18}, \qquad t = \frac{2\pi}{3} + \frac{7\pi}{18} = \frac{19\pi}{18}, \qquad t = \frac{4\pi}{3} + \frac{7\pi}{18} = \frac{31\pi}{18}$$

which also belong to the interval [0,2π].

Comment: In the previous example, as in other similar cases, two different solution formulas arise. Therefore, we need to solve two inequalities with respect to k, one for each formula. If the equation belonged to a case where the solution formulas could be unified, the procedure would be simpler, since checking a single formula would be sufficient. This is why it is useful to unify solution formulas whenever this is possible.

II) Equations using trigonometric identities

In some trigonometric equations, trigonometric numbers appear with powers or in combinations that do not allow a direct reduction to one of the basic forms. In such cases, we use appropriate trigonometric identities in order to simplify the equation and transform it into a familiar form.

Example 2: Solve the equation $ημ^2x + 5συν^2x = 4$.
Solution: We use the identity $ημ^2x + συν^2x = 1$, and the equation becomes:

$$ημ^2x + συν^2x + 4συν^2x = 4 \Leftrightarrow 1 + 4συν^2x = 4 \Leftrightarrow 4συν^2x = 3 \Leftrightarrow συν^2x = \frac{3}{4} \Leftrightarrow$$

$$συνx = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

We now solve each of the equations $συνx=\frac{\sqrt{3}}{2}$ and $συνx=-\frac{\sqrt{3}}{2}$. Since $συν\frac{π}{6}=\frac{\sqrt{3}}{2}$, we have $συν(π-\frac{π}{6})=-\frac{\sqrt{3}}{2}$. Therefore:

$$συνx = \frac{\sqrt{3}}{2} \Leftrightarrow συνx = συν\frac{π}{6} \Leftrightarrow$$
$$x = 2kπ \pm \frac{π}{6} \ (k \in \mathbb{Z})$$

$$συνx = -\frac{\sqrt{3}}{2} \Leftrightarrow συνx = συν\frac{5π}{6} \Leftrightarrow$$
$$x = 2kπ \pm \frac{5π}{6} \ (k \in \mathbb{Z})$$

Remark: If we write $x=2kπ+\frac{5π}{6}=2kπ+π-\frac{π}{6}=(2k+1)π-\frac{π}{6}$ and similarly $x=2kπ-\frac{5π}{6}=2kπ-π+\frac{π}{6}=(2k-1)π+\frac{π}{6}$, we see that in this case as well the solution formulas can be unified in the form $x=kπ\pm\frac{π}{6}$ $(k\in\mathbb{Z})$.

Example 3: Solve the equation $εφx \cdot σφ(2x) = 1$.

Solution: We use the identity $εφx \cdot σφx = 1$, therefore $εφx=\frac{1}{σφx}$, and the equation becomes:

$$\frac{1}{σφx} \cdot σφ(2x) = 1 \Leftrightarrow σφ(2x) = σφx \Leftrightarrow 2x = kπ + x \Leftrightarrow x = kπ \ (k \in \mathbb{Z})$$

However, since εφx and σφ(2x) are not defined for every real number x, we must check whether the solutions we found are acceptable. εφ(kπ) is defined, but σφ(2kπ) is not defined for any integer k. Therefore, all the solutions must be rejected, and the equation is impossible.

Example 4: Solve the equation ημx = $\sqrt{3}$συνx.
Solution: If συνx=0, then from the equation we would also have ημx=0, which is impossible, since for every angle x we have $ημ^2x + συν^2x = 1$. Therefore, συνx≠0. Dividing both sides of the equation by συνx, we obtain:

$$\frac{\text{ημx}}{\text{συνx}} = \sqrt{3} \Leftrightarrow \text{εφx} = \sqrt{3} \Leftrightarrow \text{εφx} = \text{εφ}\frac{\pi}{3} \Leftrightarrow \text{x} = \text{k}\pi + \frac{\pi}{3}\ (\text{k} \in \mathbb{Z})$$

III) Equations with change of trigonometric number (reduction to the first quadrant)

This category includes trigonometric equations in which different trigonometric numbers appear, such as ημ together with συν, or εφ together with σφ, of the same or of different angles, connected by the sign + or −. In these cases, we try to transform the trigonometric number in one side of the equation into another one, using the reduction formulas to the first quadrant, in order to obtain a basic trigonometric equation.

Example 5: Solve the equation εφx = σφ(3x).
Solution: We transform σφ into εφ using the complementary angle. Thus, the equation becomes:

$$\text{εφx} = \text{εφ}\left(\frac{\pi}{2} - 3\text{x}\right) \Leftrightarrow \text{x} = \text{k}\pi + \frac{\pi}{2} - 3\text{x} \Leftrightarrow 4\text{x} = \text{k}\pi + \frac{\pi}{2} \Leftrightarrow \text{x} = \frac{\text{k}\pi}{4} + \frac{\pi}{8}\ \ (\text{k}\epsilon\mathbb{Z})$$

We now must check whether εφx and σφ(3x) are defined for angles of this form. Since $\frac{k\pi}{4}+\frac{\pi}{8}=\frac{(2k+1)\pi}{8}$, x is an odd multiple of $\frac{\pi}{8}$, so it cannot be of the form λπ+$\frac{\pi}{2}$, which is an even multiple of $\frac{\pi}{8}$. Therefore, εφx is defined. Moreover, 3x is also an odd multiple of $\frac{\pi}{8}$, so it is always different from λπ, which is an even multiple of $\frac{\pi}{8}$. Hence, σφ(3x) is also defined, and all the solutions we found are acceptable.

Example 6: Solve the equation συν(2x) + ημx = 0.
Solution: The equation becomes συν(2x) = −ημx, that is, συν(2x) = ημ(−x). We transform ημ into συν of the complementary angle. Thus, we obtain:

$$\text{συν}(2\text{x}) = \text{συν}\left(\frac{\pi}{2} + \text{x}\right) \Leftrightarrow \begin{matrix} 2\text{x} = 2\text{k}\pi + \frac{\pi}{2} + \text{x} \Leftrightarrow \text{x} = 2\text{k}\pi + \frac{\pi}{2}\ \text{or} \\ 2\text{x} = 2\text{k}\pi - \frac{\pi}{2} - \text{x} \Leftrightarrow 3\text{x} = 2\text{k}\pi - \frac{\pi}{2} \Leftrightarrow \text{x} = \frac{2\text{k}\pi}{3} - \frac{\pi}{6} \end{matrix}\ (\text{k}\epsilon\mathbb{Z})$$

We observe that here as well the first type of solutions appears as a special case of the second one (for example, for k=1 the second formula gives x=$\frac{2\pi}{3}-\frac{\pi}{6}=\frac{\pi}{2}$). Therefore, all solutions of the equation are given by x=$\frac{2k\pi}{3}-\frac{\pi}{6}$ $(k\epsilon\mathbb{Z})$.

Exercises

1. Solve the equations:

   i) $ημx=\frac{\sqrt{2}}{2}$ ii) $συνx=\frac{\sqrt{2}}{2}$ iii) $εφx=\frac{\sqrt{3}}{3}$ iv) $σφx=1$

2. Solve the equations:

   i) $ημx=-\frac{\sqrt{3}}{2}$ ii) $συνx=-\frac{\sqrt{2}}{2}$ iii) $εφx=-\frac{\sqrt{3}}{3}$ iv) $σφx=-\frac{\sqrt{3}}{3}$

3. Solve the equations:

   i) $(1-ημx)(2ημx-\sqrt{3})=0$ ii) $(2ημx+\sqrt{2})(1-συνx)=0$ iii) $(2συνx+1)(εφ^2x-3)σφx=0$

4. Solve the equations:

   i) $ημ(x+\frac{π}{3})=-1$ ii) $2συν(3x-\frac{π}{4})=1$ iii) $εφ(\frac{π}{4}-5x)=\sqrt{3}$

5. Solve the equations:

   i) $2ημ^2ω+ημω-1=0$ ii) $2συν^2x+3συνx-2=0$ iii) $3εφ^2t=3+2\sqrt{3}εφt$

6. Find the solutions of the equation $εφx = 1$ in the interval $(3π, 4π)$.
7. Solve the equation $εφx=σφ(x+\frac{π}{3})$ in the interval $[0,2π)$.
8. Solve the equations:

   i) $ημx+συν(\frac{π}{4}-x)=0$ ii) $εφ(2x)-σφ(\frac{π}{3}+3x)=0$

   [**Hint:** Work as in Examples 5 and 6.]
9. Solve the equation $εφx \cdot ημx+1=ημx+εφx$.

   [**Hint:** Move all terms to the left-hand side and factorize. Pay attention to the restrictions.]
10. Solve the equation $\frac{1}{συν^2x}-2εφx=4$.

    [**Hint:** 1. Use the trigonometric identity $συν^2x=\frac{1}{1+εφ^2x}$.

    2. Express the final solution using an angle θ such that $εφθ=3$.]
11. Solve the equations:

    i) $ημx-συνx=0$ ii) $ημx+συνx=0$

    [**Hint:** You may work either as in Example 6 or as in Example 4.]