4.1 Introduction to Polynomials

1. The polynomial as an algebraic expression

A polynomial in the variable x is any algebraic expression of the form
$P(x)=\alpha_\nu x^\nu + \alpha_{\nu-1}x^{\nu-1} + \ldots + \alpha_1 x + \alpha_0$, where:

- ν is a natural number,
- $\alpha_0, \alpha_1, \ldots, \alpha_{\nu-1}, \alpha_\nu$ are real numbers.

In the writing of a polynomial, it is allowed that some of its coefficients are equal to zero. Therefore, a polynomial is not necessarily written in its simplest form.

The addends $\alpha_0, \alpha_1 x, \ldots, \alpha_{\nu-1}x^{\nu-1}, \alpha_\nu x^\nu$ are called the **terms** of the polynomial. Each term of a polynomial has the form $\alpha_k x^k$, where k is a natural number and α_k is a real number. Such an algebraic expression is called a **monomial** (in x).

The numbers $\alpha_0, \alpha_1, \ldots, \alpha_{\nu-1}, \alpha_\nu$ are called the **coefficients** of the polynomial.

The number α_0 is called the constant term.

Examples of polynomials:

- $2x^3 - 5x + 1$. Here we have $\nu=3$, $\alpha_0=1$, $\alpha_1=-5$, $\alpha_2=0$ because the x^2 term is missing, $\alpha_3=2$.
- $-x^2 + 4x$. Here we have $\nu=2$, $\alpha_0=0$ because the constant term is missing, $\alpha_1=4$, $\alpha_2=-1$.
- 7. Here we have $\nu=0$, $\alpha_0=7$.

A polynomial is called **constant** if it can be written in the form $P(x)= \alpha_0$. In particular, the constant polynomial $P(x)=0$ is called the **zero polynomial.**

The following algebraic expressions are not polynomials: $\frac{1}{x}$, $\sqrt{x}$, $x^{-2} + 1$. This is because, in the writing of a polynomial, it is not allowed:

- for the variable x to appear in the denominator,
- to appear under a radical,
- or to be raised to an exponent that is not a natural number.

2. Numerical value of a polynomial

Let P(x) be a polynomial and let ρ be a real number. The **numerical value** of the polynomial P(x) at x=ρ is the number obtained when we replace x by the number ρ. It is denoted by P(ρ).

Example: If $P(x) = 2x^3 - 5x + 1$, then $P(1) = 2\cdot 1^3 - 5\cdot 1 + 1= -2$.

In this way, every polynomial $P(x)=\alpha_\nu x^\nu + \alpha_{\nu-1}x^{\nu-1} + \ldots + \alpha_1 x + \alpha_0$ defines a function with domain $\mathbb{R}$, which assigns to each real number ρ the numerical value P(ρ). This function is called a **polynomial function**.

A real number ρ is called a **root** of the polynomial P(x), if P(ρ) = 0. That is, a root is any number for which the numerical value of the polynomial is zero.

Example: For the polynomial $P(x) = x^2 - 3x + 2$, the numbers 1 and 2 are roots, because P(1) = 0 and P(2) = 0.

3. Equality of polynomials

Two polynomials P(x) and Q(x) are said to be equal if all their corresponding coefficients are equal.
More specifically, if
$P(x)=\alpha_v x^v + \alpha_{v-1}x^{v-1} + \ldots + \alpha_1 x + \alpha_0$ and $Q(x) = \beta_\mu x^\mu + \beta_{\mu-1}x^{\mu-1} + \ldots + \beta_1 x + \beta_0$, where $v \le \mu$, then the polynomials P(x) and Q(x) are equal if and only if:
$\alpha_0 = \beta_0$, $\alpha_1 = \beta_1$, … , $\alpha_v = \beta_v$ and $\beta_{v+1} = \beta_{v+2} = \ldots = \beta_\mu = 0$.
In other words, for two polynomials to be equal, every term that appears in both polynomials must have the same coefficient, while any additional terms that appear only in one polynomial (if any) must have zero coefficient.

For example, the polynomials $P(x) = x^3 + 2x - 1$ and $Q(x) = 0x^4 + x^3 + 0x^2 + 2x - 1$ are equal, because all their corresponding coefficients are equal.
Also, for the polynomials $P(x) = x^2 - 2x$ and $Q(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$ to be equal, we must have $\delta=0$, $\gamma=-2$, $\beta=1$ and $\alpha=0$.

Remark: From the definition it follows that if two polynomials P(x) and Q(x) are equal, then their numerical values are equal for every real number x. The converse is also true: if P(x) = Q(x) for every $x \in \mathbb{R}$, then the polynomials P(x) and Q(x) are equal.

Convention: In what follows, we adopt the following convention:

- When we write the equality P(x) = Q(x), we mean equality of numerical values, not equality of polynomials.
- If we want to state that the polynomials P(x) and Q(x) are equal, we write: P(x) = Q(x) for every $x \in \mathbb{R}$.

4. Degree of a polynomial

Let a polynomial $P(x) = \alpha_v x^v + \alpha_{v-1}x^{v-1} + \cdots + \alpha_1 x + \alpha_0$, where v is a natural number and $\alpha_0, \alpha_1, \ldots, \alpha_v$ are real numbers.

If the polynomial is **not the zero polynomial**, then the **degree** of the polynomial P(x) is defined as the greatest natural number k for which the coefficient α_k is different from zero.

Remarks:

1. The degree is defined only **for non-zero polynomials**.
2. A polynomial may not be written in its simplest form; however, its degree is determined by the greatest exponent of x that appears **with a non-zero coefficient**.
3. Every constant and non-zero polynomial $P(x) = \alpha_0$, with $\alpha_0 \neq 0$, has degree 0.
4. If the polynomial $P(x) = \alpha_v x^v + \alpha_{v-1}x^{v-1} + \cdots + \alpha_1 x + \alpha_0$ has degree k, then it can be written in the form $P(x) = \alpha_k x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_1 x + \alpha_0$, with $\alpha_k \neq 0$.

Παραδείγματα:

The polynomial $P(x) = 2x^4 - 3x^2 + 5$ has degree 4.
The polynomial $Q(x) = 0x^5 + x^3 + 0x^2 - 7$ has degree 3.
The polynomial $R(x) = 7$ has degree 0.
The polynomial $S(x) = 0$ has no degree.
If the polynomial $p(x) = \alpha x^2 + \beta x + \gamma$ has degree 2, then $\alpha \neq 0$.

5. Operations on polynomials

We can add, subtract, or multiply polynomials by using the properties of algebraic expressions. The result of these operations is always a polynomial, as shown in the following examples:

$(x^3 + 2x^2 - 5x + 7) + (4x^2 + 5x - 3) = x^3 + 2x^2 - 5x + 7 + 4x^2 + 5x - 3 = x^3 + 6x^2 + 0x + 4 = x^3 + 6x^2 + 4$

[Here we add a polynomial of degree 3 and a polynomial of degree 2, and the sum is a polynomial of degree 3.]

$(2x^3 - x^2 + 1) + (-2x^3 + 2x - 3) = 2x^3 - x^2 + 1 - 2x^3 + 2x - 3 = 0x^3 - x^2 + 2x - 2 = -x^2 + 2x - 2$

[Here we add two polynomials of degree 3, and the sum is a polynomial of degree 2.]

$(x^3 - 3x^2 - 1) + (-x^3 + 3x^2 + 1) = x^3 - 3x^2 - 1 - x^3 + 3x^2 + 1 = 0$

[Here we add two polynomials of degree 3, and the sum is the zero polynomial, for which the degree is not defined.]

$(x^3 + 2x^2 - 5x + 7) - (4x^3 - 5x^2 + 3) = x^3 + 2x^2 - 5x + 7 - 4x^3 + 5x^2 - 3 = -3x^3 + 7x^2 - 5x + 4$

[Here we subtract two polynomials of degree 3, and the difference is a polynomial of degree 3.]

$(x^2 + 5x) \cdot (2x^3 + 3x - 1) = x^2 \cdot (2x^3 + 3x - 1) + 5x \cdot (2x^3 + 3x - 1) =$

$= 2x^5 + 3x^3 - x^2 + 10x^4 + 15x^2 - 5x = 2x^5 + 10x^4 + 3x^3 + 14x^2 - 5x$

[Here we multiply a polynomial of degree 2 by a polynomial of degree 3, and the product is a polynomial of degree 5.]

For the degree of the sum and the product of two non-zero polynomials, the following hold:

- In general, the degree of the sum or the difference of two polynomials, if it is defined, is less than or equal to the maximum of the degrees of the two addends. In particular:
  - If the two addends have different degrees, then the sum and the difference have degree equal to the maximum of the two degrees.
  - If the two addends have the same degree, then the sum or the difference may have a smaller degree, or may even be the zero polynomial.
- The degree of the product of two non-zero polynomials is always equal to the sum of the degrees of the two factors.

Example 1: Find the values of $\alpha \in \mathbb{R}$ for which the polynomials $P(x)=(\alpha^2-3\alpha)x^3+x^2+\alpha$ and $Q(x)=-2x^3+\alpha^2x^2+(\alpha^3-1)x+1$ are equal.

Solution: For two polynomials to be equal, every term that appears in both polynomials must have the same coefficient, while any additional terms that appear only in one of the polynomials must have zero coefficient. Therefore, we obtain the equations:
$\alpha^2-3\alpha=-2$, $1=\alpha^2$, $0=\alpha^3-1$ and $\alpha=1$.
The last equation is already solved with respect to α, and we see that the value α=1 also satisfies the remaining equations. Therefore, the two polynomials are equal for α=1.

Example 2: Find the real numbers α and β for which the polynomial $P(x)=3x^3+\alpha x^2+\beta x-6$ has roots −2 and 3.
Solution: We must have P(−2)=0 and P(3)=0. Substituting, we obtain the equations:
$3(-2)^3+\alpha(-2)^2+\beta(-2)-6=0 \Leftrightarrow -24+4\alpha-2\beta-6=0 \Leftrightarrow 4\alpha-2\beta=30 \Leftrightarrow 2\alpha-\beta=15$
$3\cdot 3^3+\alpha\cdot 3^2+\beta\cdot 3-6=0 \Leftrightarrow 81+9\alpha+3\beta-6=0 \Leftrightarrow 9\alpha+3\beta=-75 \Leftrightarrow 3\alpha+\beta=-25$
We solve the system using the method of opposite coefficients. The coefficients of β in the two equations are opposite, so by adding the equations member by member we obtain 5α=−10, therefore **α=−2**. Substituting, for example, into the second equation, we find −6+β=−25, hence **β=−19**.

Example 3: Find the degree of the polynomial $P(x)=(9\lambda^3-4\lambda)x^3+(9\lambda^2-4)x-3\lambda+2$ for the different values of $\lambda\in\mathbb{R}$.
Solution: We examine when the coefficient of the power of x with the greatest exponent, here x^3, is equal to zero.

$$9\lambda^3-4\lambda=0 \Leftrightarrow \lambda(9\lambda^2-4)=0 \Leftrightarrow \lambda(3\lambda-2)(3\lambda+2)=0 \Leftrightarrow \lambda=0 \text{ ή } \lambda=\frac{2}{3} \text{ ή } \lambda=-\frac{2}{3}$$

Therefore:

I) If λ≠0 and $\lambda\neq\frac{2}{3}$ and $\lambda\neq-\frac{2}{3}$, then $9\lambda^3-4\lambda\neq0$, so the polynomial P(x) is of degree 3.

II) If λ=0, by substitution we find $P(x)=0x^3+(-4)x+2=-4x+2$, which is of degree 1.

III) If $\lambda=\frac{2}{3}$, we find $P(x)=0x^3+0x-3\cdot\frac{2}{3}+2=0$, which is the zero polynomial, and its degree is not defined.

IV) If $\lambda=-\frac{2}{3}$, we find $P(x)=0x^3+0x-3\cdot(-\frac{2}{3})+2=4$, which is of degree 0.

Example 4: Find the real numbers α, β, γ for which the polynomial $P(x)=3x^2-7x+5$ can be written in the form $P(x)=\alpha x(x+1)+\beta x+\gamma$.
Solution: We start from the form $P(x)=\alpha x(x+1)+\beta x+\gamma$ and proceed step by step:

$$P(x)=\alpha x(x+1)+\beta x+\gamma=\alpha x^2+\alpha x+\beta x+\gamma= \alpha x^2+(\alpha+\beta)x+\gamma$$

In order for this polynomial to be equal to $P(x)=3x^2-7x+5$, we must have:
α=3, α+β=−7, γ=5. Therefore, α=3, β=−10 and γ=5.

Remark: The same method was used in the second solution of **Example 4** in **§2.2**.

Exercises

1. Which of the following expressions are polynomials in x?

   i) $1-x^3$ ii) $\alpha^3-3\alpha^2x+3\alpha x^2-x^3$ ($\alpha\in\mathbb{R}$) iii) $x+\frac{1}{x}$ iv) $x^4-2x^{\frac{1}{3}}+4x-1$

   For those that are not polynomials, justify your answer.

2. Examine which of the given numbers are roots of the following polynomials:

   i) $P(x)=2x^3-3x^2+2x+7$, $x=-1$, $x=1$

   ii) $Q(x)=-x^4+1$, $x=-1$, $x=1$, $x=3$

3. Find for which values of $k\in\mathbb{R}$ the number 2 is a root of the polynomial $P(x)=x^3-kx^2+5x+k$.

4. For which values of $\alpha\in\mathbb{R}$, is the value of the polynomial $P(x)=5x^2+3\alpha x+\alpha^2-2$ equal to 1 when $x=-1$?

5. Find the real numbers λ and μ for which the polynomial $P(x)=2x^3+\lambda x^2+\mu x+6$ has 1 as a root and satisfies $P(-2)=-12$.

6. i) Find for which values of $\mu\in\mathbb{R}$ the polynomial $P(x)=(4\mu^3-\mu)x^3+4(\mu^2-\frac{1}{4})x-2\mu+1$ is the zero polynomial.

   ii) For all remaining values of μ, determine the degree of the polynomial.

7. The polynomials $P(x)=x^2-5x+2$ and $Q(x)=x^3+3x+1$ are given.

   A. Without performing the calculations, find the degree of each of the following polynomials:

      i) $P(x)+Q(x)$ ii) $2P(x)-3Q(x)$ iii) $P(x)\cdot Q(x)$ iv) $[P(x)]^2$

   B. Verify your conclusions by computing the above polynomials.

8. Find a polynomial P(x) such that $(2x+1)P(x)=2x^3-9x^2-3x+1$ for all $x\in\mathbb{R}$.

   [**Hint:** First show that P(x) must be a polynomial of degree 2. Then assume $P(x)=\alpha x^2+\beta x+\gamma$ and determine the values of α, β, γ so that the polynomials are equal, as in Example 4.]