4.2 Division of Polynomials

1. The division identity and the division algorithm

Just as we do with numbers, we can also divide one polynomial by another, provided that the divisor is not the zero polynomial.

Before we proceed, it is important to observe the following:

the sum, the difference, and the product of two polynomials are always polynomials. On the contrary, the division of two polynomials, when we consider them simply as algebraic expressions, does not in general result in a polynomial.

For example, the division $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ gives a polynomial, whereas the division $\frac{x^2-1}{x}$ gives $x - \frac{1}{x}$, and this last algebraic expression is not a polynomial.

A similar situation occurs with integers. The sum, the difference, and the product of two integers are integers, while the quotient of two integers is not necessarily an integer. For this reason, when we speak about division of integers, we do not mean a simple fraction, but a process that leads to a quotient and a remainder.

Recall that if Δ (“Διαιρετέος”, the dividend) and δ (“διαιρέτης”, the divisor) are two integers with $\delta \neq 0$, then there exist two unique integers π (“πηλίκο”, the quotient) and υ (“υπόλοιπο”, the remainder) such that:

$$\Delta=\delta\cdot\pi+\upsilon \quad \text{and} \quad 0\leq\upsilon<\delta$$

For example, in the division:

95 −7 25 −21 4	7 13

We have Δ=95 and δ=7.

We first divided 9 by 7 and wrote the first digit of the quotient, which is 1. Then we multiplied 1 by the divisor and subtracted it from 9. We brought down the second digit of the dividend, 5. Next, we divided the number 25 that was formed by 7 to find the next digit of the quotient.

Thus, we finally obtained π=13 and υ=4. We verify that:

$95=7\cdot13+4$ and $0\leq4<7$, so the division is complete.

Using the same idea, we define the division of polynomials. The goal of the division is to write the original polynomial as the product of the divisor and another polynomial, called the quotient, plus a remainder.

More precisely, the following Polynomial Division Theorem holds:

Theorem: If P(x) and δ(x) are two polynomials with δ(x) not the zero polynomial, then there exist two polynomials π(x) and υ(x) such that:

- $P(x)=\delta(x)\cdot\pi(x)+\upsilon(x)$ (division identity)
- υ(x) is the zero polynomial or has degree less than the degree of δ(x).

As in the division of integers, P(x) is called the **dividend**, δ(x) the **divisor**, π(x) the **quotient**, and υ(x) the **remainder** of the division.

Example 1: Perform the division $2x^4-3x^2+x-1$ by x^2-2x+3.

Solution: We use the division layout and follow the steps below:

1. We write the dividend and the divisor in the division layout. In the dividend, we leave empty places for missing terms (here, the x^3 term).

$$
\begin{array}{rrrrr|l}
2x^4 & & -3x^2 & +x & -1 & x^2-2x+3 \\ \cline{6-6}
\underline{-2x^4} & \underline{+4x^3} & \underline{-6x^2} & & & 2x^2+4x-1 \\
 & 4x^3 & -9x^2 & +x & -1 & \\
 & \underline{-4x^3} & \underline{+8x^2} & \underline{-12x} & & \\
 & & -x^2 & -11x & -1 & \\
 & & \underline{+x^2} & \underline{-2x} & \underline{+3} & \\
 & & & -13x & +2 &
\end{array}
$$

2. We divide the leading terms of the dividend and the divisor and write the first term of the quotient. Then we multiply this term by all the terms of the divisor and subtract the result from the dividend.
3. We check whether the result of the subtraction is the zero polynomial or has degree smaller than the degree of the divisor. If yes, the division is completed. Otherwise, we repeat Step 2.

In our example, the first term of the quotient is $2x^4:x^2=2x^2$. The result of the first subtraction is a polynomial of degree 3, so we continue. The second term of the quotient is $4x^3:x^2=4x$. The result of the second subtraction has degree 2, equal to the degree of the divisor, so we continue. The third term of the quotient is $-x^2:x^2=-1$. The result of the third subtraction is a polynomial of degree 1, smaller than the degree of the divisor, so the division is completed.

Therefore, $\pi(x)= 2x^2+4x-1$ and $\upsilon(x)=-13x+2$. The division identity is:
$2x^4-3x^2+x-1=(x^2-2x+3)(2x^2+4x-1)-13x+2$.

If the remainder of a division is the zero polynomial, then:

- The division is called **exact** ("τέλεια").
- P(x) **is divisible by** ("διαιρείται"), or **has as a divisor** ("έχει διαιρέτη"), or **has as a factor** ("έχει παράγοντα") δ(x).
- δ(x) **divides** ("διαιρεί"), or is a **divisor** ("είναι διαιρέτης"), or is a **factor** ("είναι παράγοντας") of P(x).

Example 2: In Exercise 8 of §4.1, we were looking for a polynomial P(x) such that $(2x+1)P(x)= 2x^3-9x^2-3x+1$ for every $x\in\mathbb{R}$. Another solution can be obtained by dividing the polynomial $2x^3-9x^2-3x+1$ by $2x+1$.

Solution (2nd method):

$$
\begin{array}{rrrr|l}
2x^3 & -9x^2 & -3x & +1 & 2x+1 \\ \cline{5-5}
\underline{-2x^3} & \underline{-x^2} & & & \\
 & -10x^2 & -3x & +1 & x^2-5x+1 \\
 & \underline{+10x^2} & \underline{+5x} & & \\
 & & 2x & +1 & \\
 & & \underline{-2x} & \underline{-1} & \\
 & & & 0 &
\end{array}
$$

The division gives quotient x^2-5x+1 and remainder 0.
Therefore, the division identity is:
$2x^3-9x^2-3x+1 =(2x+1)(x^2-5x+1)$, and thus $P(x)=x^2-5x+1$.

Example 3: Polynomial division can also be used in **Example 4** of §4.1: Find the real numbers α, β, γ such that the polynomial $P(x)=3x^2-7x+5$ can be written in the form $P(x)=\alpha x(x+1)+\beta x+\gamma$.

Solution: This can be solved by dividing P(x) by $x(x+1)=x^2+x$. Try the division in the scheme below and write the division identity.

$3x^2 \quad -7x \quad +5$	x^2+x
....	....
....	

The division identity gives:
.... Therefore α=..., β=..., γ=....

2. Division by x−ρ

When the divisor in a polynomial division is a polynomial of the form $x-\rho$, the following theorems hold. They allow us to determine information about the result of the division without actually performing it:

Theorem 1: The remainder of the division of a polynomial P(x) by $x-\rho$ is equal to the numerical value of the polynomial P(x) at $x=\rho$. That is, $\upsilon=P(\rho)$.

Proof: Since the divisor has degree 1, the remainder is either the zero polynomial or a polynomial of degree 0, that is, a constant polynomial.
Let $\upsilon(x)=\upsilon$. Then the division identity is written: $P(x)=(x-\rho)\cdot\pi(x)+\upsilon$. Substituting $x=\rho$, we obtain: $P(\rho)=(\rho-\rho)\cdot\pi(\rho)+\upsilon=0\cdot\pi(\rho)+\upsilon=\upsilon$. Therefore, $P(\rho)=\upsilon$.
Conclusion: The division identity for $P(x):(x-\rho)$ is written:

$$P(x)=(x-\rho)\cdot\pi(x)+P(\rho)$$

Theorem 2: A polynomial P(x) has $x-\rho$ as a factor if and only if ρ is a root of P(x), that is, if and only if $P(\rho)=0$.

Proof: P(x) has factor $x-\rho \Leftrightarrow$ the division $P(x):(x-\rho)$ is exact, that is, the remainder is 0 $\Leftrightarrow$ $P(\rho)=0 \Leftrightarrow$ ρ is a root of P(x).

Example 4: Find the remainder of the division $(18x^{80}-6x^{50}+4x^{20}-2) : (x+1)$.

Solution: The divisor is $x+1=x-(-1)$. By Theorem 1, the remainder is $\upsilon=P(-1)=18(-1)^{80}-6(-1)^{50}+4(-1)^{20}-2=18-6+4-2=14$.

Example 5: Find the values of k for which $x-1$ is a factor of the polynomial $P(x) = k^2x^4 + 3kx^2 - 4$.

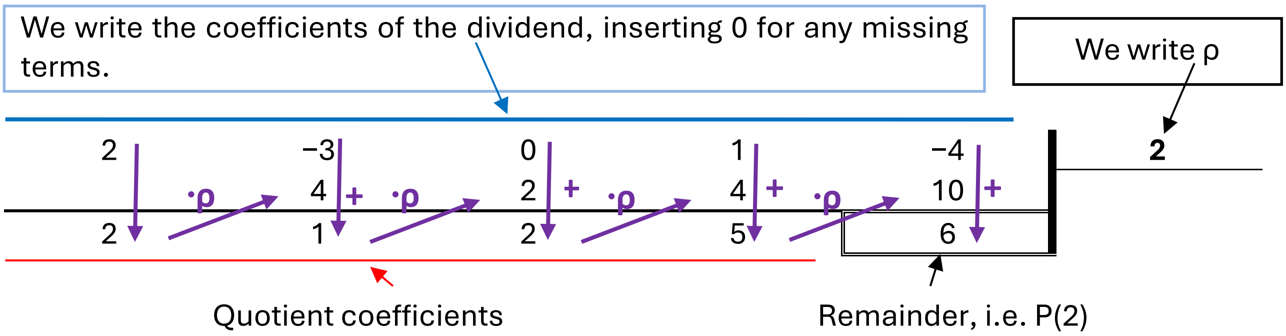
Solution: For $x-1$ to be a factor, we must have $P(1)=0 \Leftrightarrow k^2+3k-4=0$. Solving the quadratic equation, we find $k=1$ or $k=-4$.

3. Horner's Scheme

Horner's scheme is an easy and fast method for performing a polynomial division when the divisor is of the form $x-\rho$.

For example, to perform the division $(2x^4-3x^3+x-4):(x-2)$ we construct the following table:

- In the first row of the table, we write the coefficients of the dividend P(x), inserting 0 for any missing terms, and on the far right we write ρ.
- We bring down the first coefficient to the third row.
- We multiply by ρ and write the result in the second row, one position to the right, under the coefficient of the next term.
- We add the numbers of the first and second rows and write the result in the third row.
- We repeat the last two steps until all coefficients of the dividend have been used.
- When the process is completed, the rightmost entry of the third row is the remainder of the division, that is P(ρ), while the remaining entries of the third row, from left to right, are the coefficients of the quotient π(x).



In our example, we find $\pi(x)=2x^3+x^2+2x+5$, $\upsilon=P(2)=6$, and the division identity is written: $2x^4-3x^3+x-4=(x-2)(2x^3+x^2+2x+5)+6$.

Example 6: Perform the division of the polynomial $P(x) = 3x^3 + 6x^2 - 17x + 20$ by $x+3$.

Solution: The divisor is $x+3=x-(-3)$, and the Horner scheme is:

3	+6	−17	+20	**−3**
	−9	+9	+24	
3	−3	−8	+44	

Therefore, the remainder is $\upsilon=P(-3)=44$, the quotient is $\pi(x)=3x^2-3x-8$ and the division identity is: $3x^3+6x^2-17x+20=(x+3)(3x^2-3x-8)+44$.

Example 7: Prove that $x-1-\sqrt{3}$ is a factor of the polynomial $P(x) = x^3 - 3x^2 + 2$.

Solution: The divisor is $x-1-\sqrt{3}=x-(1+\sqrt{3})$, and the Horner scheme is:

1	−3	0	+2	**$1+\sqrt{3}$**
	$1+\sqrt{3}$	$1-\sqrt{3}$[1]	−2[2]	
1	$-2+\sqrt{3}$	$1-\sqrt{3}$	0	

Since the remainder is $\upsilon=0$, $x-1-\sqrt{3}$ is a factor of the polynomial $P(x) = x^3-3x^2+2$.

1 On scratch paper, we perform the multiplication: $(-2+\sqrt{3})(1+\sqrt{3}) = -2-2\sqrt{3}+\sqrt{3}+3 = 1-\sqrt{3}$.

2 Also, $(1-\sqrt{3})(1+\sqrt{3})=1-(\sqrt{3})^2 = 1-3 = -2$.

In many cases, a polynomial division problem may require more than one successive division by divisors of the form $x-\rho$. In the following examples, we will apply such techniques.

Example 8: Prove that the polynomial $P(x) = 2x^4 - 6x^3 + 5x^2 - 3x + 2$ is divisible by $(x-1)(x-2)$ and find the quotient.

Solution: First, we divide $P(x)$ by $x-1$ and then divide the quotient of the first division by $x-2$. The first division is performed using the Horner scheme as follows:

2	−6	5	−3	2	**1**
	2	−4	1	−2	
2	−4	1	−2	0	

Thus, $P(x)=(x-1)(2x^3-4x^2+x-2)$. We continue with the second division:

2	−4	1	−2	**2**
	4	0	2	
2	0	1	0	

Hence, $2x^3-4x^2+x-2=(x-2)(2x^2+1)$, therefore $P(x)=(x-1)(x-2)(2x^2+1)$. Thus, $(x-1)(x-2)$ is a factor of $P(x)$ and the quotient is $2x^2+1$.

Example 9: Find the real numbers α, β for which the polynomial $P(x) = \alpha x^{\nu+1} + \beta x^{\nu} + 1$ has $(x-1)^2$ as a factor.

Solution: First, $P(x)$ must have $x-1$ as a factor $\Leftrightarrow P(1)=0 \Leftrightarrow \alpha+\beta+1=0 \Leftrightarrow \underline{\beta=-\alpha-1}$.
Then $P(x)=\alpha x^{\nu+1} + (-\alpha-1)x^{\nu} + 1$. We divide $P(x)$ by $x-1$. Since the degree of the dividend is $\nu+1$ (undetermined), we complete the missing intermediate terms with ... and take care with the counting.

α	−α−1	0	...	0	1	**1**
	α	−1	...	−1	−1	
α	−1	−1	...	−1	0	

Thus $\upsilon=0$. The degree of the quotient is one less than the degree of the dividend, so the quotient has degree ν.
Hence $\pi(x)=\alpha x^{\nu}-x^{\nu-1}-x^{\nu-2}-\ldots-x-1$ and $P(x)=(x-1)\pi(x)$.
For $(x-1)^2$ to be a factor of $P(x)$, $x-1$ must also be a factor of $\pi(x) \Leftrightarrow \pi(1)=0 \Leftrightarrow$
$\alpha \underbrace{-1-1-\cdots-1}_{\nu \text{ times}} = 0 \Leftrightarrow \alpha - \nu = 0 \Leftrightarrow \alpha = \nu$ and therefore $\beta=-\nu-1$. Then $\pi(x)=(x-1)q(x)$, where $q(x)$ is the quotient of $\pi(x)$ divided by $x-1$. Thus, $P(x)=(x-1)^2q(x)$, so $(x-1)^2$ is a factor of $P(x)$.

Exercises

1. Perform the following divisions and write the division identity in each case.
   i) $(24x^5+20x^3-12x^2-15) : (6x^2+5)$
   ii) $(2x^4+4x^3-5x^2+3x-2) : (x^2+2x-3)$
2. Using the Horner scheme, find the quotients and the remainders of the following divisions:
   i) $(-x^3+75x-250) : (x+10)$

ii) $(4x^3+16x^2-23x-15) : (x+\frac{1}{2})$

3. If $P(x)=-2x^3-2x^2-x+2409$, use the Horner scheme to find $P(-11)$.
4. Prove that the polynomials of the form $x-\rho$ given in each case are factors of $P(x)$.

i) $P(x)=x^4-25x^2+144$, $x+3$

ii) $P(x)=16x^4-8x^3+9x^2+14x-4$, $x-\frac{1}{4}$

5. Prove that the following polynomials do not have a factor of the form $x-\rho$.

i) $P(x)=4x^4+7x^2+12$ ii) $Q(x)=-5x^6-3x^2-4$

[**Hint:** Prove that they have no real root ρ.]

6. Perform the following divisions:

i) $(3x^2-2\alpha x-8\alpha) : (x-2\alpha)$ ii) $(x^3+\alpha x^2-\alpha^2 x-\alpha^3) : (x+\alpha)$

7. Prove that the polynomial $P(x)=(x+1)^{2v}-x^{2v}-2x-1$, where v is a positive integer, has as factors all the factors of $2x^3+3x^2+x$.

[**Hint:** Factorize $2x^3+3x^2+x = 2x(x+1)(x+\frac{1}{2})$. Then show that each of the polynomials $x=x-0$, $x+1$, $x+\frac{1}{2}$ is a factor of $P(x)$.]