4.3 Polynomial Equations and Inequalities

I. Polynomial Equations

The basic strategy we follow in order to solve a polynomial equation is factorization:

1. We move all terms to the left-hand side. On the right-hand side we leave only 0.
2. We factor the polynomial on the left-hand side of the equation.
3. We set each factor equal to 0.

That is, if we arrive at the equation P(x) = 0,
and we find that P(x) = (factor 1) · (factor 2) · … · (factor k),
then the equation is equivalent to:
(factor 1) = 0 or (factor 2) = 0 or … or (factor k) = 0.
Therefore, if all the factors are of first or second degree, then the equation can be solved completely. This is our goal: **to factor the polynomial on the left-hand side into first- or second-degree factors**.

In the factorization process we use the known methods:

- common factor,
- known identities,
- grouping of terms,
- splitting a term.

In addition to these, we can also use the following result:

From **Theorem 2** of **§4.2**, if ρ is a root of the polynomial P(x), then x − ρ is a factor of P(x). Therefore, if we find a root ρ, we can factor P(x) by dividing P(x) by x − ρ (for example, using Horner's scheme).

Thus, if we cannot factor using the known techniques, we search for roots of the polynomial P(x). In this search, the following Theorem is very helpful:

Theorem (Integer Roots Theorem): If the polynomial $P(x)=\alpha_\nu x^\nu + \alpha_{\nu-1}x^{\nu-1} + \dots + \alpha_1 x + \alpha_0$ has integer coefficients, then its possible integer roots are the divisors of the constant term α_0.

Example 1: Solve the equation $x^3+2x^2-9x=18$.

Solution: We move all terms to the left-hand side: $x^3+2x^2-9x-18=0$.[1] The possible integer roots are the divisors of −18, namely ±1, ±2, ±3, ±6, ±9, ±18. By testing, we find that −2 is a root of $x^3+2x^2-9x-18$, so we perform the division using Horner's scheme.

[1] The polynomial on the left-hand side can also be factored by grouping: $x^2(x+2)-9(x+2)=0$, etc.

1	+2	−9	−18	**−2**
	−2	0	+18	
1	0	−9	0	

Therefore, $x^3+2x^2-9x-18 = (x+2)(x^2-9)$ and the equation is equivalently written as:

$$(x+2)(x^2-9)=0 \Leftrightarrow x=-2 \text{ ή } x^2=9 \Leftrightarrow x=-2 \text{ ή } x=\pm 3$$

Example 2: Solve the equation $x^3+x^2-2=0$. [2]

Solution: The possible integer roots are the divisors of −2, namely ±1, ±2. By testing, we find that 1 is a root of x^3+x^2-2, so we perform the division using Horner's scheme.

1	+1	0	−2	**1**
	1	2	+2	
1	2	2	0	

Therefore, $x^3+x^2-2 = (x-1)(x^2+2x+2)$ and the equation is equivalently written as:

$$(x-1)(x^2+2x+2)=0 \Leftrightarrow x=1 \text{ ή } x^2+2x+2=0$$

The quadratic equation $x^2+2x+2=0$ has discriminant $\Delta=2^2-4\cdot 1\cdot 2=4-8=-4<0$, therefore it has no real roots. Thus, the only real root of the equation is $x=1$.

Example 3: Solve the equation $x^4-3x^3+6x-4=0$.

Solution: The possible integer roots are the divisors of −4, namely ±1, ±2, ±4. By testing, we find that 1 is a root of x^4-3x^3+6x-4, so we perform the division using Horner's scheme.

1	−3	0	+6	−4	**1**
	+1	−2	−2	+4	
1	−2	−2	+4	0	

Thus, the equation can be written equivalently as $(x-1)(x^3-2x^2-2x+4)=0$. The possible integer roots of x^3-2x^2-2x+4 are the divisors of 4, namely ±1, ±2, ±4. By testing, we find that 2 is a root of x^3-2x^2-2x+4, and we continue with Horner's scheme for the second division.

1	−2	−2	+4	**2**
	+2	0	−4	
1	0	−2	0	

Therefore, $x^3-2x^2-2x+4 = (x-2)(x^2-2)$ and the original equation becomes equivalently:

$$(x-1)(x-2)(x^2-2)=0 \Leftrightarrow x=1 \text{ or } x=2 \text{ or } x^2=2 \Leftrightarrow x=1 \text{ or } x=2 \text{ or } x=\pm\sqrt{2}$$

Example 4: Let $P(x)=x^6-5x^4-10x^2+k$. Find the values of k for which $x-1$ is a factor of $P(x)$. Then solve the equation $P(x)=0$.

Solution: For $x-1$ to be a factor of $P(x)$, we must have $P(1)=0$. Substituting $x=1$: $1-5-10+k=0$, therefore $k=14$. Thus, the equation $P(x)=0$ becomes: $x^6-5x^4-10x^2+14=0$.

[2] The polynomial in the left-hand side can also be factorized using term splitting and identities: $x^3-1+x^2-1=0 \Leftrightarrow (x-1)(x^2+x+1)+(x-1)(x+1)=0$, etc.

Set $x^2=\omega$. Then we obtain the equation: $\omega^3-5\omega^2-10\omega+14=0$. Since $x = 1$ is a root of P(x), after the substitution $\omega = x^2$ we get $\omega=1^2=1$ as a root of the new equation. We perform the division using Horner's scheme:

1	−5	−10	+14	**1**
	+1	−4	−14	
1	−4	−14	0	

Thus, the equation can be written equivalently as:

$$(\omega-1)(\omega^2-4\omega-14)=0 \Leftrightarrow \omega=1 \text{ or } \omega^2-4\omega-14=0.$$

The quadratic equation has discriminant $\Delta=16+56=72$. Hence,

$$\omega=1 \text{ or } \omega=\frac{4\pm\sqrt{72}}{2}=\frac{4\pm6\sqrt{2}}{2}=2\pm3\sqrt{2}.$$

Since $\omega=x^2$, we obtain:

$x^2=1$ or $x^2=2+3\sqrt{2}$ or $x^2=2-3\sqrt{2} \Leftrightarrow x=\pm1$ or $x=\pm\sqrt{2+3\sqrt{2}}$, and the last equation has no real solutions, because $2-3\sqrt{2}<0$.

II. Polynomial Inequalities

We proceed as in equations. We move all terms to the left-hand side, factor into linear and quadratic factors, find the roots of each factor, and then construct a sign chart in order to solve the inequality.

Recall that a quadratic factor $\alpha x^2+\beta x+\gamma$ is:

- Always of the same sign as α if the discriminant Δ satisfies $\Delta<0$, that is, if it has no roots.
- Zero at exactly one point (double root) and elsewhere of the same sign as α, if $\Delta=0$.
- Having two roots and between them it has the opposite sign of α, while outside the roots it has the same sign as α, if $\Delta>0$.

Example 5: Solve the inequality $(x^3-x^2+2x-2)(x^2-9)>0$.

Solution: The factor x^3-x^2+2x-2 has constant term 2 and possible integer roots ±1, ±2. By testing, we find that 1 is a root of the polynomial, so we factor using Horner's scheme:

1	−1	+2	−2	**1**
	+1	0	+2	
1	0	+2	0	

Thus, the inequality becomes: $(x-1)(x^2+2)(x^2-9)>0$. We find the roots of each factor:

$x-1=0 \Leftrightarrow x=1$, $x^2+2=0 \Leftrightarrow x^2=-2$, impossible in IR $x^2-9=0 \Leftrightarrow x^2=9 \Leftrightarrow x=\pm3$.

Sign chart:

x		−3		1		3	
x−1	−		−	0	+		+
x^2+2	+		+		+		+
x^2-9	+	0	−		−	0	+
Product	−	0	+	0	−	0	+

Therefore, $(x-1)(x^2+2)(x^2-9)>0$ for $x\in(-3,1)\cup(3,+\infty)$.

Example 6: Solve the inequality $x^3-4x^2-3x+18\leq0$.

Solution: The constant term is 18, so the possible integer roots are ±1, ±2, ±3, ±6, ±9, ±18. By testing, we find that −2 is a root of the polynomial, so we factor using Horner's scheme:

1	−4	−3	+18	**−2**
	−2	+12	−18	
1	−6	+9	0	

Thus, the inequality becomes: $(x+2)(x^2-6x+9) \leq 0$. We find the roots of each factor:
$x+2=0 \Leftrightarrow x=-2$, $x^2-6x+9=0 \Leftrightarrow x=3$ (double root), since $\Delta=(-6)^2-4\cdot 1\cdot 9=36-36=0$.
Sign chart:

x		−2		3	
x+2	−	0	+		+
x^2-6x+9 [3]	+		+	0	+
Product	−	0	+	0	+

Therefore, the solution of the inequality $(x+2)(x^2-6x+9) \leq 0$ is $x\in(-\infty, -2]\cup\{3\}$, that is, $x\leq -2$ or $x=3$.

Example 7: Find the intervals on which the graph of the polynomial function $f(x)=x^5+6x^4+9x^3-4x^2-12x$ lies above the x-axis.

Solution: We must have $f(x)>0$ and, factoring out x as a common factor, we obtain equivalently: $x(x^4+6x^3+9x^2-4x-12)>0$. The constant term of the fourth-degree factor is −12, so the possible integer roots are ±1, ±2, ±3, ±4, ±6, ±12. By testing, we find that 1 is a root, and we factor using Horner's scheme:

1	+6	+9	−4	−12	**1**
	+1	+7	+16	+12	
1	+7	+16	+12	0	

Thus, the inequality becomes: $x(x-1)(x^3+7x^2+16x+12)>0$. The constant term of the cubic factor is 12, and the possible integer roots are ±1, ±2, ±3, ±4, ±6, ±12. Since all its coefficients are positive, it cannot have positive roots, so we test only the negative ones. We find that −2 is a root and continue with Horner's scheme for the second division:

1	+7	+16	+12	**−2**
	−2	−10	−12	
1	+5	+6	0	

Therefore, the inequality becomes: $x(x-1)(x+2)(x^2+5x+6)>0$. The roots are: $x=0$, $x=1$, $x=-2$ for the linear factors, while the quadratic factor has discriminant $\Delta=25-24=1$ and roots $x=\frac{-5\pm\sqrt{1}}{2}$, that is $x=-2$ and $x=-3$.
We construct the sign table:

---

[3] Here we observe that the factor x^2-6x+9, although it becomes zero at 3, does not change sign to the left and to the right of 3, because the root is double.
A similar situation may occur with the sign of a product, if a number makes more than one of its factors equal to zero.
Therefore, we must not change the sign mechanically at every root; instead, each time we must examine the sign of the factor or of the total product.

x	−3	−2	0	1	
x	−	−	− 0	+	+
x−1	−	−	−	− 0	+
x+2	−	− 0	+	+	+
x^2+5x+6	+ 0	− 0	+	+	+
Product [4]	− 0	+ 0	+ 0	− 0	+

Therefore, the solution of the inequality $x(x-1)(x+2)(x^2+5x+6)>0$ is $x\in(-3,-2)\cup(-2,0)\cup(1,+\infty)$.

Exercises

1. Solve the following equations.
   i) $3x^5+5x^4=3x^3+5x^2$
   ii) $x^3-7x+6=0$
   iii) $x^3-3x^2+x+2=0$
   iv) $3x^3+8x^2-15x+4=0$
   v) $x^3-10x-12=0$
   vi) $x^3+2x^2+7x+6=0$
2. Prove that the following equations have no integer roots.
   i) $x^4+3x-2=0$
   ii) $2x^4-3x^3+6x^2-24x+5=0$
3. Solve the following inequalities.
   i) $2x^5-162x\leq 0$
   ii) $2x^3-5x^2-6x+9>0$
   iii) $x^4-6x^3+22x^2-30x+13\leq 0$
   iv) $x^3-3x+2<0$
4. Find the common points of the graph of each of the following functions with the x-axis:
   i) $f(x)=3x^3-3x^2-5x-2$
   ii) $g(x)=4x^3-3x-1$
5. Find the intervals on which the graph of the polynomial function $f(x)=x^4-5x^3+3x^2+x$ lies below the x-axis.
6. Solve the following equations.
   i) $\frac{1}{10}x^3+\frac{1}{2}x^2+\frac{1}{5}x-\frac{4}{5}=0$
   ii) $x^3-\frac{5}{6}x^2-\frac{22}{3}x+\frac{5}{2}=0$

   [**Hint:** Multiply by appropriate numbers in order to obtain polynomials with integer coefficients.]
7. Find the values of $\alpha, \beta\in\mathbb{R}$ such that the polynomial $P(x)= x^4+\alpha x^3+\beta x^2-16x-12$ has $x+1$ and $x-2$ as factors. Then solve the equation $P(x)=0$.
8. Find the values of $k\in\mathbb{Z}$ for which the equation $x^3-x^2+kx+3=0$ has at least one integer root.

   [**Hint:** Since $k\in\mathbb{Z}$, the possible integer roots are the divisors of 3. Test each one to determine the corresponding value of k. The resulting values of k must be integers.]

---

[4] Here we have the case mentioned in the previous footnote: at −2 the product becomes zero, but it does not change sign. This happens because −2 is a root of two different factors of the product.
If we factor the quadratic factor x^2+5x+6 into $(x+2)(x+3)$, we see that the product can also be written in the form $x(x-1)(x+2)^2(x+3)$. Therefore, in this case as well, −2 is a double root of the product.