4.4 Equations and Inequalities Reducible to Polynomial Form

In this section we will study three types of equations and one type of inequalities which, through appropriate transformations, can be converted into polynomial equations or inequalities. The way this conversion is carried out depends on the form of the given equation or inequality. However, in all cases we follow the same basic strategy:

- We determine the restrictions, that is, the values of the unknown for which the equation or inequality is defined (when required).
- We transform the expression into polynomial form.
- We solve the resulting polynomial equation or inequality.
- Finally, we check the solutions.

Below, we present the method and illustrative examples for each category of equations and inequalities that we will examine.

I. Rational Equations

In this category belong equations in which the unknown appears in the denominator of a fraction. Such an equation is defined only when all denominators are different from zero. The solution procedure is the following:

- We factor, whenever possible, the denominators and find their least common multiple (LCM).
- We set LCM ≠ 0 and determine the **restrictions**.
- We eliminate the denominators, thus transforming the equation into a polynomial equation.
- We solve the resulting polynomial equation, as in the previous section.
- Finally, we check which solutions satisfy the restrictions.

Attention: We deal with the restrictions only at the beginning and at the end, not in the intermediate steps. It is possible that some of the potential integer roots of the polynomial equation are rejected due to the restrictions. We do not ignore them during the factorization process — we examine them normally. Rejection takes place exclusively in the final step.

Example 1: Solve the equation $\frac{3x^2-1}{x-1} - \frac{2}{x^2-x} = \frac{x^2-3x+2}{x}$

Solution: The equation can be written equivalently as $\frac{3x^2-1}{x-1} - \frac{2}{x(x-1)} = \frac{x^2-3x+2}{x}$. Therefore, the LCM is $x(x-1)$ and we obtain the restrictions: $\boxed{LCM = x(x-1) \neq 0, \text{ i.e. } x \neq 0 \text{ and } x \neq 1}$

We proceed by eliminating the denominators:

$$x(x-1)\frac{3x^2-1}{x-1}-x(x-1)\frac{2}{x(x-1)}=x(x-1)\frac{x^2-3x+2}{x}$$

$$3x^3-x-2=x^3-3x^2+2x-x^2+3x-2$$

We bring all terms to the left-hand side and solve the resulting polynomial equation:

$$2x^3+4x^2-6x=0 \Leftrightarrow 2x(x^2+2x-3)=0 \Leftrightarrow x=0 \text{ or } x^2+2x-3=0$$

The quadratic equation has discriminant $\Delta=4+12=16$, and roots $x=1$ or $x=-3$.
Thus, the roots are $x=0$ (rejected due to the restrictions), $x=1$ (rejected due to the restrictions) and $x=-3$ (accepted).

II. Equations with radicals

This category includes equations in which the unknown appears under a radical sign (square root, cube root, or higher index root). Such an equation is defined only when all expressions under radicals are greater than or equal to 0. We must also remember that the value of a radical is itself greater than or equal to 0. If the equation contains only one radical, the solving procedure is as follows:

- We keep one radical alone on one side of the equation. Any other terms that appear with it are moved to the other side.
- We determine the **restrictions**: everything under a radical, and everything that is set equal to a radical, must be ≥ 0. We solve and combine the inequalities to obtain the final restriction.
- We raise both sides of the equation to the second power, the third power, or to whatever power is required in order to eliminate the radical.

> **Attention:** Squaring does not always preserve equivalence of equations; it does so only if both sides are non-negative. See also the activity "Are the equations equivalent?".

- If necessary, we repeat the above steps until all radicals are eliminated.
- We solve the resulting polynomial equation.
- Finally, we check whether the solutions obtained satisfy the restrictions.

If the equation contains two or more radicals, we adjust the first steps of the procedure (isolation and restrictions) in such a way that, before raising to a power, we ensure that both sides of the equation are non-negative. If this cannot be guaranteed in advance, then after solving we must verify that the obtained solutions satisfy the original equation.

Example 2: Solve the equation $\sqrt{2x+7}-x=2$.

Solution: The equation is written equivalently as: $\sqrt{2x+7}=x+2$. We determine the restrictions: $2x+7\geq 0 \Leftrightarrow x\geq -\frac{7}{2}$ και $x+2\geq 0 \Leftrightarrow x\geq -2$. Combining them, we obtain: $\boxed{x\geq -2}$

We now square both sides:

$$2x+7=(x+2)^2 \Leftrightarrow 2x+7=x^2+4x+4 \Leftrightarrow x^2+2x-3=0$$

Solving the quadratic equation, we find: $x=1$ (accepted) or $x=-3$ (rejected due to the restrictions).

Example 3: Solve the equation $\sqrt{1+2\sqrt{x}}=\sqrt{x+1}$.

Solution: We determine the restrictions: $x\geq 0$ and $x+1\geq 0 \Leftrightarrow x\geq -1$. Combining them, we obtain $\boxed{x\geq 0.}$ Then it also follows that $1+2\sqrt{x}\geq 0$.

Squaring both sides:

$$1+2\sqrt{x}=x+1 \Leftrightarrow 2\sqrt{x}=x$$

From this equation, it also follows that $x\geq 0$, which has already been taken into account. We continue by squaring again:

$$4x=x^2 \Leftrightarrow 4x-x^2=0 \Leftrightarrow x(4-x)=0 \Leftrightarrow x=0 \text{ or } x=4.$$

Both solutions satisfy the restriction $x\geq 0$, therefore they are accepted.

Example 4: Solve the equation $\sqrt{x-1}+\sqrt{2x-1}=x$.

Solution: We determine the restrictions: $x-1\geq 0 \Leftrightarrow x\geq 1$ and $2x-1\geq 0 \Leftrightarrow x\geq \frac{1}{2}$. Combining them, we obtain $\boxed{x\geq 1.}$ Then for the right-hand side we also have $x\geq 0$, so no additional restriction is needed.

Squaring both sides:

$$\left(\sqrt{x-1}\right)^2+2\sqrt{x-1}\cdot\sqrt{2x-1}+\left(\sqrt{2x-1}\right)^2=x^2 \Leftrightarrow$$

$$x-1+2\sqrt{x-1}\cdot\sqrt{2x-1}+2x-1=x^2 \Leftrightarrow 2\sqrt{(x-1)(2x-1)}=x^2-3x+2$$

We determine restrictions again: If $x\geq 1$, then $2x-1\geq 0$, so the radical on the left-hand side is defined. We must also have $x^2-3x+2\geq 0$. Since the roots of the trinomial are 1 and 2, this means $x\leq 1$ or $x\geq 2$. Combining with the initial restriction $x\geq 1$, we obtain $\boxed{x=1 \text{ ή } x\geq 2.}$

Since the quadratic polynomial x^2-3x+2 has roots 1 and 2, it can be factorized as $x^2-3x+2=(x-1)(x-2)$. We now square both sides again, and solve the resulting polynomial equation by factorization.

$$\begin{aligned} 4(x-1)(2x-1)=(x-1)^2(x-2)^2 &\Leftrightarrow (x-1)(8x-4)-(x-1)^2(x-2)^2=0 \\ &\Leftrightarrow (x-1)[8x-4-(x-1)(x^2-4x+4)]=0 \\ &\Leftrightarrow (x-1)(8x-4-x^3+4x^2-4x+x^2-4x+4)=0 \\ &\Leftrightarrow (x-1)(-x^3+5x^2)=0 \Leftrightarrow -x^2(x-1)(x-5)=0 \end{aligned}$$

Thus we find: $x=0$ (rejected due to restrictions), or $x=1$ (accepted), or $x=5$ (accepted).

III. Equations Solved by Substitution

This category includes equations in which the same expression appears repeatedly (e.g. ημx, x^2, x^2+1, etc.). In such cases:

- We introduce a new variable (e.g. ω) equal to the repeated expression.
- We solve the resulting polynomial equation.
- We return to the original variable and solve the equation obtained with respect to it.

To solve the polynomial equation, we use the techniques we have already developed.

Example 5: Solve the equation $6\left(\frac{x}{x+1}\right)^2+5\left(\frac{x}{x+1}\right)-6=0$.

Solution: The equation is defined for $x+1\neq0$, that is $x\neq-1$. Let $\omega=\frac{x}{x+1}$. Then the equation becomes: $6\omega^2+5\omega-6=0$. The discriminant is $\Delta=25+144=169$, therefore the roots of the quadratic equation are $\omega=\frac{-5\pm\sqrt{169}}{2\cdot6}=\frac{-5\pm13}{12}$, that is, $\omega=\frac{8}{12}=\frac{2}{3}$ or $\omega=\frac{-18}{12}=-\frac{3}{2}$. Returning to the original variable x, we obtain:

$$\frac{x}{x+1}=\frac{2}{3}\Leftrightarrow 3x=2x+2\Leftrightarrow x=2$$

or

$$\frac{x}{x+1}=-\frac{3}{2}\Leftrightarrow 2x=-3x-3\Leftrightarrow$$

$$5x=-3\Leftrightarrow x=-\frac{3}{5}$$

Both solutions satisfy the restriction $x\neq-1$, therefore they are accepted.

Example 6: Solve the equation $2\eta\mu^3x+\sigma\upsilon\nu^2x+2\eta\mu x-2=0$.

Solution: From the trigonometric identity $\eta\mu^2x+\sigma\upsilon\nu^2x=1$ (**§3.2**), we obtain $\sigma\upsilon\nu^2x=1-\eta\mu^2x$, so the equation becomes: $2\eta\mu^3x+1-\eta\mu^2x+2\eta\mu x-2=0$.
Let $\eta\mu x=\omega$. Then we obtain the polynomial equation $2\omega^3-\omega^2+2\omega-1=0$. This equation has no integer roots[1], but it is easily solved by factorization:

$\omega^2(2\omega-1)+2\omega-1=0\Leftrightarrow(2\omega-1)(\omega^2+1)=0\Leftrightarrow\omega=\frac{1}{2}$ or $\omega^2=-1$, which is impossible.

Therefore, $\eta\mu x=\frac{1}{2}\Leftrightarrow\eta\mu x=\eta\mu\frac{\pi}{6}\Leftrightarrow x=2k\pi+\frac{\pi}{6}$ or $x=2k\pi+\pi-\frac{\pi}{6}=2k\pi+\frac{5\pi}{6}$ $(k\in\mathbb{Z})$. [See **§3.5**.]

IV. Rational Inequalities

This category includes inequalities in which the unknown appears in the denominator of a fraction. Such an inequality is defined only when all denominators are different from zero.

Attention: Unlike equations, in rational inequalities it is generally <u>not allowed</u> to eliminate denominators. This is permitted only if the least common multiple (LCM) of the denominators is certainly positive, for example if it is $(x^2+1)(x^2+4)$ or something similar.

[1] Indeed, the possible integer roots are the divisors of −1, namely ±1, and by direct substitution we verify that none of them satisfies the equation.

Therefore, the method for solving a rational inequality is modified as follows:

- We move all terms to the left-hand side, so that 0 remains on the right-hand side.
- We express the fractions on the left-hand side with a common denominator, without eliminating denominators.
- We factorize wherever possible and the inequality takes the form $\frac{f(x)}{g(x)}$ ≥0 or ≤0.
- With the **restriction** g(x)≠0, the last inequality is equivalent to f(x)g(x)≥0 or ≤0, since the sign of a quotient is the same as the sign of the corresponding product.
- We solve the resulting polynomial inequality using a sign chart, as in the previous section.

Attention: If simplification occurs on the left-hand side of the inequality, we determine the restrictions before performing the simplification.

Attention: In the sign chart, on the row of x we mark not only the roots of each factor, but also the values excluded by the restrictions!

Example 7: Solve the inequality: $x^2 + \frac{2}{2x-1} \geq \frac{1}{x(2x-1)}$

Solution: We move all terms to the left-hand side, express the fractions with a common denominator, and factorize:

$$x^2 + \frac{2}{2x-1} - \frac{1}{x(2x-1)} \geq 0 \Leftrightarrow \frac{x^3(2x-1) + 2x - 1}{x(2x-1)} \geq 0 \Leftrightarrow \frac{(2x-1)(x^3+1)}{x(2x-1)} \geq 0$$

We determine the restrictions: x(2x−1)≠0, that is x≠0 and $x \neq \frac{1}{2}$. We now simplify the factor 2x−1 and factorize x^3+1 using the sum of cubes identity. The inequality becomes:

$$\frac{(x+1)(x^2-x+1)}{x} \geq 0$$

We convert the quotient into a product:

$$x(x+1)(x^2-x+1) \geq 0$$

We find the roots of each factor:

x=0, x+1=0⟺x=−1,

and the trinomial x^2-x+1 has discriminant Δ=−3<0, therefore it has no real roots.

We now construct the sign chart. We mark 0 and also $\frac{1}{2}$, which arise from the restrictions. (The restricted values are marked with double hatching in the sign chart).

x		−1		0		$\frac{1}{2}$	
x	−		−	‖	+	‖	+
x+1	−	0	+	‖	+	‖	+
x^2-x+1	+		+	‖	+	‖	+
Product	+	0	−	‖	+	‖	+

From the sign chart we conclude that the product is ≥0 for $x \in (-\infty, -1] \cup (0, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$.

Example 8: Solve the inequality: $\frac{1}{x^2} \geq \frac{x}{x^2+4}$.

Solution: Here we may proceed as before. However, in this case the least common multiple of the denominators is LCM=$x^2(x^2+4)\geq 0$, so we are allowed to eliminate denominators. In general, this is rare, but whenever it is possible it considerably simplifies the work.
We determine the restriction: LCM=$x^2(x^2+4)\neq 0$, that is $x\neq 0$. Since for every $x\neq 0$ we have $x^2(x^2+4)>0$, we multiply both sides by the LCM and obtain:

$$x^2(x^2+4)\frac{1}{x^2} \geq x^2(x^2+4)\frac{x}{x^2+4} \Leftrightarrow x^2+4\geq x^3 \Leftrightarrow x^3-x^2-4\leq 0$$

The possible integer roots are: ±1, ±2, ±4. By testing, we find that 2 is a root, and we factorize using Horner's scheme.

1	–1	0	–4	**2**
	2	2	4	
1	1	2	0	

Therefore, the inequality becomes:

$$(x-2)(x^2+x+2)\leq 0$$

We find the roots of each factor:
$x-2=0 \Leftrightarrow x=2$, and the trinomial x^2+x+2 has $\Delta=-7<0$, therefore it has no real roots.
Sign chart:

x		0	2	
x–2	–	–	0	+
x^2+x+2	+	+		+
Product	–	–	0	+

The sign chart shows that the solution of the inequality is $x\in(-\infty, 0)\cup(0, 2]$.

Exercises

1. Solve the equation: $\frac{x^2}{x-1} - \frac{2}{x+1} = \frac{4}{x^2-1}$.
2. Solve the equations:
   i) $\sqrt{x^3} = -4x$ ii) $\sqrt{3x-2} = 4$ iii) $\sqrt{5x-1} = -4$
   iv) $\sqrt{x+3} = x+1$ v) $\sqrt{x+3} = \sqrt{10-x} + 1$ vi) $\sqrt{x} + \sqrt{x-20} = 10$

   [**Hint:** Equations (i) and (iii) are solved directly using the restrictions, while (ii) is straightforward. For (iv), proceed as in **Example 2**, and for (v) and (vi) as in **Example 4**.]

3. Solve the inequalities:
   i) $\frac{x^2-x-2}{x^2+x-2} \leq 0$ ii) $\frac{2x+3}{x-1} > 4$ iii) $\frac{x^2-3x-10}{x-1} + 2 \leq 0$ iv) $\frac{x}{2x-1} \geq \frac{3}{x+2}$
4. Solve the equations:
   i) $x + 3\sqrt{x} - 10 = 0$ ii) $\sqrt[3]{x^2} + \sqrt[3]{x} - 6 = 0$ iii) $x^2 + x - 4 = \sqrt{x^2+x-2}$

[**Hint:** Equation (i) can be solved as in **Example 2**, but also using the substitution $\omega=\sqrt{x}$. In (ii) use the substitution $\omega=\sqrt[3]{x}$, and in (iii) use the substitution $\omega=x^2+x$.]

5. Solve the equation: $\sqrt{x-1}+\sqrt{x-4}=\sqrt{x+4}$.
6. Solve the equation: $2\eta\mu^4x-3\eta\mu^3x-3\sigma\upsilon\nu^2x-3\eta\mu x+4=0$.