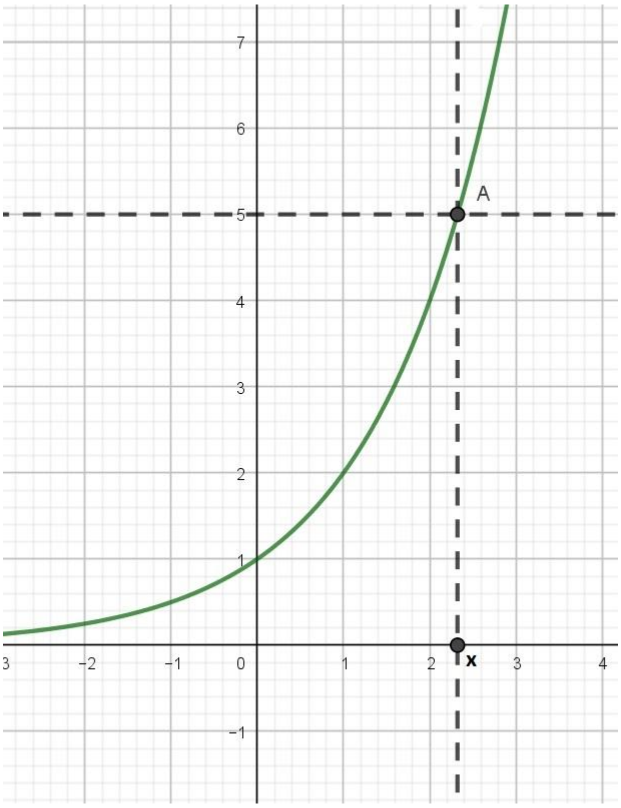
5.2 The Concept of the Logarithm

So far, with the help of the properties of exponential functions, we have seen that we can solve equations of the form $3^x=81$, $5^x=\frac{1}{25}$, $2^{x+1}=\frac{\sqrt{2}}{8}$, $4^x=8^{2-x}$, etc. However, if we attempt to solve an equation such as $2^x=5^{1-x}$, or even the seemingly simpler $2^x=5$, we encounter a difficulty. The problem is that, unlike the previous cases, we cannot express 2 as a power with base 5, nor 5 as a power with base 2, nor both 2 and 5 as powers with the same base. The solution to this problem will be provided by the concept of the logarithm, which we define in the following section.

1. Definition of the Logarithm



Let us consider, for example, the equation $2^x=5$. Graphically, we can see that this equation has a unique solution, as follows: We draw the graph of the exponential function $y=2^x$ and, from the point 5 on the y-axis, we draw the line $y=5$, which is parallel to the x-axis. This line intersects the graph at exactly one point. If x is the abscissa of the point of intersection, it follows immediately from this construction that this value of x is the unique solution of the equation $2^x=5$. We then write: $x=\log_2 5$, which is read as “the logarithm of 5 to base 2.”.

In a similar way, we can prove that if $0<\alpha\neq 1$ and $\theta>0$, then the equation $\alpha^x=\theta$ has a unique solution, denoted by $\log_\alpha\theta$. Therefore,

$$\boxed{x=\log_\alpha\theta \Leftrightarrow \alpha^x=\theta, \text{ where } \theta>0 \text{ and } 0<\alpha\neq 1}$$

Thus, we are led to the following definition:

Definition: If $0<\alpha\neq 1$ and $\theta>0$, the logarithm of θ to base α is the exponent to which α must be raised in order to obtain θ. It is denoted by $\log_\alpha\theta$.

Examples: $\log_2 8=3$, since $2^3=8$. $\log_4 2=\frac{1}{2}$, since $4^{\frac{1}{2}}=\sqrt{4}=2$.

$\log_{10} 0.001=-3$, since $10^{-3}=0.001$. $\log_{0.5} 0.25=2$, since $(0.5)^2=0.25$.

From the definition of the logarithm, the following properties arise immediately:

$$\log_\alpha \alpha^x = x \qquad \alpha^{\log_\alpha \theta} = \theta \qquad \log_\alpha 1 = 0 \qquad \log_\alpha \alpha = 1$$

Two important special cases of logarithms are the following:

- **Common Logarithm (Decimal Logarithm):** The logarithm with base 10 is called the decimal logarithm. We simply write $\log \theta$ instead of $\log_{10}\theta$. From the above, it is clear that: $x = \log \theta \Leftrightarrow 10^x = \theta$, where θ>0.

$$\log 10^x = x \qquad 10^{\log \theta} = \theta \qquad \log 1 = 0 \qquad \log 10 = 1$$

- **Natural (Napierian) Logarithm[1]:** The logarithm with base e is called the natural or Napierian logarithm. We write $\ln \theta$ instead of $\log_e\theta$. It is also evident that: $x = \ln \theta \Leftrightarrow e^x = \theta$, where θ>0.

$$\ln e^x = x \qquad e^{\ln \theta} = \theta \qquad \ln 1 = 0 \qquad \ln e = 1$$

2. Properties of Logarithms

In addition to the properties that arise directly from the definition, the following properties also hold for logarithms.

Theorem: If α, β are positive numbers different from 1, θ_1, θ_2, θ>0, and k∈IR, then:

1. $\log_\alpha(\theta_1\theta_2) = \log_\alpha \theta_1 + \log_\alpha \theta_2$
2. $\log_\alpha \frac{\theta_1}{\theta_2} = \log_\alpha \theta_1 - \log_\alpha \theta_2$
3. $\log_\alpha \theta^k = k \log_\alpha \theta$
4. $\log_\beta \theta = \frac{\log_\alpha \theta}{\log_\alpha \beta}$ (Change-of-Base Formula).

Proof: 1. Let $x_1 = \log_\alpha \theta_1$ and $x_2 = \log_\alpha \theta_2$. Then, by the definition of the logarithm, $\alpha^{x_1} = \theta_1$ and $\alpha^{x_2} = \theta_2$. Therefore, $\theta_1\theta_2 = \alpha^{x_1} \cdot \alpha^{x_2} = \alpha^{x_1+x_2}$. Again, by the definition of the logarithm, the last equality is equivalent to

$$\log_\alpha(\theta_1\theta_2) = x_1 + x_2 = \log_\alpha \theta_1 + \log_\alpha \theta_2$$

and the result is proved.

2. It is proved similarly.

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[1] The term Napierian logarithm derives from the name of the Scottish mathematician John Napier (1550-1617), who first published logarithmic tables in 1614. If you wonder why these logarithms, based on the irrational number *e*, are also called natural logarithms — what could be natural about logarithms with such a base? — you will see in the third grade that they arise naturally when one attempts to calculate the area enclosed by an arc of a hyperbola and three straight line segments.

3. Let x = $\log_\alpha \theta$. Then, by definition, $\alpha^x = \theta$. Hence, $\theta^k = (\alpha^x)^k = \alpha^{kx}$. Again, by the definition of the logarithm, this equality is equivalent to
$\log_\alpha \theta^k = kx = k\log_\alpha \theta$, and the proof is complete.

4. The proof is omitted as it is beyond the required material.

Example 1: Evaluate the expression:

$$A = \frac{1}{2}\log_2 256 + 2\log_2 3 - \log_2 18$$

Solution: Proceeding step by step, we obtain:

$$A = \log_2 256^{\frac{1}{2}} + \log_2 3^2 - \log_2 18 = \log_2 \sqrt{256} + \log_2 9 - \log_2 18 = \log_2 16 + \log_2 \frac{9}{18} =$$

$$= \log_2\left(16 \cdot \frac{1}{2}\right) = \log_2 8 = 3$$

since 8=2^3.

Example 2: Evaluate the expression $9^{\frac{1}{2}\log_3 18 - 1}$

Solution: Proceeding step by step:

$$9^{\frac{1}{2}\log_3 18-1} = 9^{\log_3\sqrt{18}-\log_3 3} = (3^2)^{\log_3\frac{\sqrt{18}}{3}} = 3^{2\log_3\frac{3\sqrt{2}}{3}} = 3^{2\log_3\sqrt{2}} = \left(3^{\log_3\sqrt{2}}\right)^2 = \left(\sqrt{2}\right)^2 = 2$$

Second Method:

$$9^{\frac{1}{2}\log_3 18-1} = (3^2)^{\frac{1}{2}\log_3 18-1} = 3^{\log_3 18-2} = 3^{\log_3 18-\log_3 9} = 3^{\log_3\frac{18}{9}} = 3^{\log_3 2} = 2$$

Example 3: Solve the equations: i) $2^x = 5$, ii) $2^x=5^{1-x}$.

Solution: i) From the definition of the logarithm, we have: $2^x = 5 \Leftrightarrow x = \log_2 5$.
ii) We take logarithms of both sides with respect to the same base, for example base 10, and obtain successively:

$$\log 2^x = \log 5^{1-x} \Leftrightarrow x\log 2 = (1-x)\log 5 \Leftrightarrow x\log 2 = \log 5 - x\log 5 \Leftrightarrow$$
$$x\log 2 + x\log 5 = \log 5 \Leftrightarrow x(\log 2 + \log 5) = \log 5 \Leftrightarrow x\log 10 = \log 5 \Leftrightarrow x = \log 5$$

Remark: The same result could have been obtained if we had taken logarithms with respect to any other base. For example, using natural logarithms instead of common logarithms, we would obtain: $x\ln 10 = \ln 5 \Leftrightarrow x = \frac{\ln 5}{\ln 10} \Leftrightarrow x = \log 5$, by the change-of-base formula.

Example 4: According to a law of photometry, the intensity I of sunlight that penetrates vertically into seawater, or into another transparent medium, decreases exponentially as a function of the depth h (in meters), according to the function $I = I_0 \cdot e^{-\lambda h}$, where $\lambda > 0$ is a constant and $I_0 > 0$ is the initial intensity.
a) Is there a depth h at which the intensity of sunlight becomes zero?
b) It is known that for pure seawater $\lambda = 1.4$, and that a certain form of plant life cannot exist when the intensity of sunlight becomes less than or equal to one-fourth of the

initial intensity. Determine the values of the depth h for which this occurs. (Given that $\ln 2 = 0.7$).

Solution: a) Suppose that at some depth h the intensity is zero, that is, $I = 0$. Then: $I_0 \cdot e^{-\lambda h} = 0$. Since $I_0 > 0$, it would follow that $e^{-\lambda h} = 0$, which is impossible. Therefore, the intensity I can never become zero.
b) For $\lambda = 1.4$, we have $I = I_0 \cdot e^{-1.4h}$. For the specific form of plant life to exist at depth h, the following must hold:

$$I \geq \frac{1}{4} I_0 \Leftrightarrow I_0 \cdot e^{-1.4h} \geq \frac{1}{4} I_0 \Leftrightarrow e^{-1.4h} \geq \frac{1}{4} \Leftrightarrow e^{-1.4h} \geq e^{\ln\frac{1}{4}} \Leftrightarrow e^{-1.4h} \geq e^{-\ln 4}$$

Hence,

$$-1.4h \geq -\ln 4 \Leftrightarrow h \leq \frac{\ln 2^2}{1.4} \Leftrightarrow h \leq \frac{2\ln 2}{1.4}$$

Using the approximation $\ln 2 = 0.7$, we find $h \leq 1$. Therefore, this form of plant life can exist at a depth of at most 1 meter.

Exercises

1. Compute, without using a calculator, the following logarithms:
   i) $\log 0.001$ ii) $\log_{\frac{1}{10}} \sqrt{10}$ iii) $\log_9 \frac{\sqrt{27}}{3}$ iv) $\log_{\sqrt{2}} 16$

   [**Hint:** Use the equivalence $x = \log_\alpha \theta \Leftrightarrow \alpha^x = \theta$ and solve the resulting exponential equations.]

2. Find the value of x such that:
   i) $\log x = 3$ ii) $\log_4 x =-\frac{1}{2}$

   [**Hint:** Use the equivalence $x = \log_\alpha \theta \Leftrightarrow \alpha^x = \theta$.]

3. Prove that:
   i) $\log_2 3 + 2\log_2 4 - \log_2 12 = 2$ ii) $3\log 2 + \log 5 - \log 4 = 1$
   iii) $\frac{1}{2}\log 25 + \frac{1}{3}\log 8 - \frac{1}{5}\log 32 = 1 - \log 2$ iv) $2^{\log_2 6 - 2\log_2 \sqrt{3}} = 2$
   v) $2\log_2(2+\sqrt{2}) + \log_2(6-4\sqrt{2}) = 2$ vi) $4^{1-\frac{1}{2}\log_2 3} = \frac{4}{3}$

   [**Hint:** Work as in **Examples 1** and **2**.]

4. Solve the equation $3^{x-1} = 2^{x+1}$.

5. Stars are classified according to their apparent brightness into categories called magnitudes. The faintest stars, with brightness L_0 are said to have magnitude 6. Any other star with brightness L has magnitude m given by the formula: $m = 6 - 2.5 \cdot \log \frac{L}{L_0}$.
   i) Find the magnitude m of a star with brightness $L = \sqrt[5]{100} \cdot L_0$.
   ii) How many times brighter is a star of magnitude 1 than a star of magnitude 6?