5.3 The Logarithmic Function

If $0<\alpha\neq 1$, then, as we have seen previously, for every $x>0$ the logarithm $\log_\alpha x$ is defined. Therefore, the **logarithmic function with base α** is defined as $f:(0,+\infty)\to \mathbb{R}$, where $f(x)=\log_\alpha x$.

Next, we will study two specific logarithmic functions: the function $f(x)=\ln x$ (with base e) and the function $h(x)=\log x$ (with base 10). Their study will be based on the corresponding exponential functions examined previously, namely $g(x)=e^x$ and $\phi(x)=10^x$.

1. Connection Between the Logarithmic and the Exponential Function

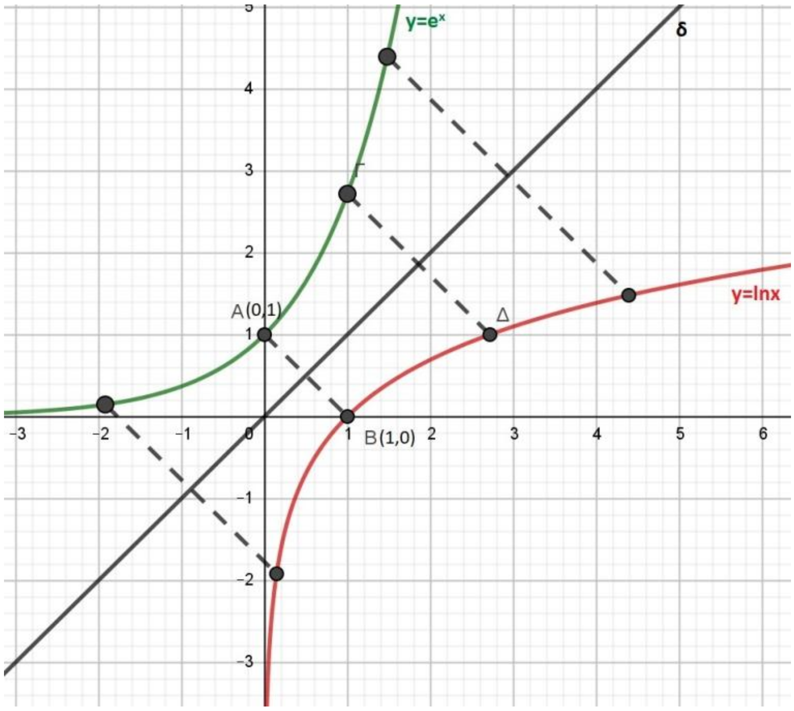
From the definition of the logarithm, we obtain the following equivalences:

$$\text{The point } M(x,y) \text{ lies on the graph of the function } f(x)=\ln x \Leftrightarrow y=\ln x \Leftrightarrow$$
$$\Leftrightarrow x=e^y \Leftrightarrow \text{the point } N(y,x) \text{ lies on the graph of } g(x)=e^x.$$

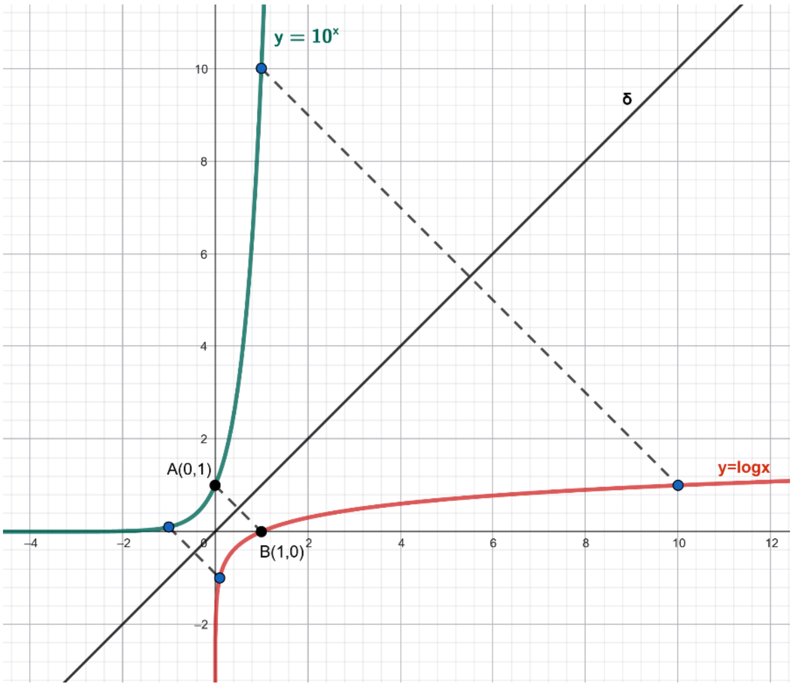
Since the points M(x,y) and N(y,x) are symmetric with respect to the bisector of the angle $x\hat{O}y$, that is, the line δ: y=x, the above equivalence implies that:

> The graphs of the functions $f(x)=\ln x$ and $g(x)=e^x$ are symmetric with respect to the line δ: y=x, which is the bisector of the angle $x\hat{O}y$.

Utilizing this symmetry, we can construct the graph of the function $f(x)=\ln x$.



In the same way, we can prove the corresponding result for the graphs of the functions h(x)=logx and ϕ(x)=10^x: they are also symmetric with respect to the line δ: y=x.[1] The graph of the function h(x)=logx is shown in the figure below.



2. Properties of the Logarithmic Function

From their graphs, the following **properties of the logarithmic functions** f(x)=lnx and h(x)=logx are derived:

- They have domain (0,+∞).[2]
- They have range IR, since for every y∈IR, y=f(e^y) and y=h(10^y).[3]
- Their graphs intersect the x-axis at the point B(1,0) and have the negative semi-axis Oy' as an asymptote.
- They are strictly increasing functions. That is, if $x_1<x_2$, then $\ln x_1<\ln x_2$ and $\log x_1<\log x_2$, and conversely. Therefore:

$$\ln x_1 < \ln x_2 \Leftrightarrow 0 < x_1 < x_2 \quad \text{and} \quad \log x_1 < \log x_2 \Leftrightarrow 0 < x_1 < x_2 \qquad \textbf{(1)}$$

- They are one-to-one functions. That is, if $x_1 \neq x_2$, then $\ln x_1 \neq \ln x_2$ and $\log x_1 \neq \log x_2$, or equivalently, if $\ln x_1=\ln x_2$ or $\log x_1=\log x_2$, then $x_1=x_2$. Hence, the following

1 As you will learn in the 3rd grade, two functions f and g for which the equivalence y=f(x) ⟺ x=g(y) holds (with appropriate restrictions on their domains) are called inverse functions, and their graphs are symmetric with respect to the line δ: y=x. Based on what we have studied so far, the functions f(x)=lnx and g(x)=e^x are inverse to each other, as are the functions h(x)=logx and ϕ(x)=10^x. Other examples of inverse functions are p(x)=x^2 (x≥0) and q(x)=$\sqrt{x}$ (x≥0), as well as many others.

2 Therefore, if a logarithm appears in an algebraic expression, equation, or inequality, we must impose restrictions: every quantity inside a logarithm must be positive.

3 This may seem surprising when looking only at the graphs, especially that of h(x)=logx, since logarithms increase at very slow rates. For example, in order for h(x) to take the value 3, x must be equal to 1000, while for it to take the value 10, x must be equal to 10^{10}=10,000,000,000. However, both functions continue to increase to infinity. More on this in the 3rd grade.

equivalences hold:

$$\ln x_1 = \ln x_2 \Leftrightarrow x_1 = x_2 \quad \text{and} \quad \log x_1 = \log x_2 \Leftrightarrow x_1 = x_2, \qquad \textbf{(2)}$$

These results hold under the condition that x_1 and x_2 are positive.

The equivalences **(1)** are used to solve logarithmic inequalities, and the equivalences **(2)** are used to solve logarithmic equations. In such inequalities and equations, we must not forget to impose **restrictions**, as mentioned in footnote [2] above, since every quantity inside a logarithm must be greater than 0.

The equivalences **(2)** were also used in **Example 3** of **§5.2** to solve exponential equations in which it was not possible to express both sides as powers with the same base.

Example 1: Solve the equation: $\log(x+1) + \log x = 1 - \log 5$

Solution: We first determine the restrictions. We must have: $x+1>0 \Leftrightarrow x>-1$ and $x>0$. Taking both conditions into account, we obtain $\boxed{x>0}$.
Next, using the properties of logarithms and equivalence **(2)**, the equation becomes:

$$\log(x+1) + \log x = \log 10 - \log 5 \Leftrightarrow \log[(x+1)\cdot x] = \log\frac{10}{5} \Leftrightarrow \log(x^2+x) = \log 2$$

$$\Leftrightarrow x^2 + x = 2 \Leftrightarrow x^2 + x - 2 = 0$$

Solving the quadratic equation, we find $x=1$ (acceptable) or $x=-2$ (rejected).

Example 2: Solve the equation: $\log\sqrt{x} = \sqrt{\log x}$

Solution: We first determine the restrictions arising from the logarithms. We must have $\boxed{x>0}$. Then $\sqrt{x}>0$ as well, and the equation becomes:

$$\log x^{\frac{1}{2}} = \sqrt{\log x} \Leftrightarrow \frac{1}{2}\log x = \sqrt{\log x}$$

Let $\log x = \omega$. The equation then takes the form: $\omega = 2\sqrt{\omega}$. The restrictions for this equation involving radicals, as discussed in **§4.4**, give the additional condition $\boxed{\omega\geq 0}$.
Under these conditions, we square both sides:

$$\omega^2 = \left(2\sqrt{\omega}\right)^2 \Leftrightarrow \omega^2 = 4\omega \Leftrightarrow \omega^2 - 4\omega = 0 \Leftrightarrow \omega(\omega - 4) = 0$$

Hence, $\omega=0$ or $\omega=4$, and both solutions satisfy the restriction on ω. Therefore, $\log x = 0 \Leftrightarrow x = 10^0 = 1$ or $\log x = 4 \Leftrightarrow x = 10^4 = 10000$. Both solutions are acceptable.

Example 3: Solve the inequality: $\log(x^2 - 4) < \log(3x)$

Solution: We first determine the restrictions. We must have $x^2 - 4 > 0$, which is equivalent to $x<-2$ or $x>2$, and also $3x>0$, that is, $x>0$. Combining these conditions, we obtain: $\boxed{x>2}$.
Next, using equivalence **(1)** and taking into account the restrictions, we get:

$$\log(x^2 - 4) < \log(3x) \Leftrightarrow x^2 - 4 < 3x \Leftrightarrow x^2 - 3x - 4 < 0$$

The quadratic expression $x^2 - 3x - 4$ has roots −1 και 4. Therefore, the inequality holds for x∈(−1,4). Combining this with the restriction x>2, we obtain the final solution: x∈(2,4).

Example 4: Prove that the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ is odd.

Solution: For the function f to be defined, it must hold that: $x + \sqrt{x^2+1} > 0$. Indeed, $x + \sqrt{x^2+1} > x + \sqrt{x^2} > x + |x| \geq x - x = 0$ for every x∈IR, since $|x| \geq -x$. Therefore, the domain of f is all real numbers IR. For the function to be odd, as we saw in **§2.1**, , it must satisfy for every x∈IR: $f(-x) = -f(x)$, or equivalently,

$$f(-x) + f(x) = 0 \Leftrightarrow \ln\left(-x + \sqrt{(-x)^2+1}\right) + \ln\left(x + \sqrt{x^2+1}\right) = 0$$

$$\Leftrightarrow \ln\left(-x + \sqrt{x^2+1}\right) + \ln\left(x + \sqrt{x^2+1}\right) = 0$$

$$\Leftrightarrow \ln\left[\left(-x + \sqrt{x^2+1}\right)\left(x + \sqrt{x^2+1}\right)\right] = 0 \Leftrightarrow \ln\left[\left(\sqrt{x^2+1}\right)^2 - x^2\right] = 0$$

$$\Leftrightarrow \ln(x^2 + 1 - x^2) = 0 \Leftrightarrow \ln 1 = 0$$

which is true. Hence, the function f is defined for all x∈IR and is odd.

Exercises

1. Solve the equations:
   i) $\log(x+1) + \log(x-1) = \log 2$ ii) $\log(x^2+1) - \log x = \log 2$

   [**Hint:** Work as in **Example 1**.]

2. Solve the equations:
   i) $\log x^2 = (\log x)^2$ ii) $(\ln x)^4 - 5(\ln x)^2 + 4 = 0$

   [**Hint:** Use the substitutions $\log x$ = ω and $(\ln x)^2$ = ω.]

3. Solve the inequality: $\log x^2 > (\log x)^2$

4. Find the domain of the function $f(x) = \ln\frac{1-x}{1+x}$ and prove that it is odd.

5. Let A = log(x−2) + log(x+2).
   i) For which values of x is A defined?
   ii) Prove that A=log(x^2−4).
   iii) Solve the equation log(x−2) + log(x+2) = 2log2 + log3.

6. Let the function $f(x) = \ln\left(\frac{e^{2x}-1}{e^x+5}\right)$.
   i) Find the domain of f.
   ii) Solve the equation f(x) = 2ln2.
   iii) Find for which values of x the graph of f lies above the x-axis.