

Θέμα 2ο (πράξεις συναρτήσεων)

Να βρείτε τις συναρτήσεις $f+g$, $f \cdot g$, $\frac{f}{g}$, $\frac{g}{f}$ στις παρακάτω περιπτώσεις:

α) $f(x) = \frac{x-1}{x-2}$, $g(x) = \ln x$, β) $f(x) = \begin{cases} x-1, & x \leq 0 \\ x, & x > 0 \end{cases}$, $g(x) = \begin{cases} x, & x \leq 1 \\ x-1, & x > 1 \end{cases}$

β) $D_f = \mathbb{R}$, $D_g = \mathbb{R}$ $D_f \cap D_g = \mathbb{R}$

i) $D_{f+g} = D_f \cap D_g = \mathbb{R}$ $(f+g)(x) = f(x) + g(x) = \begin{cases} x-1+x, & x \leq 0 \\ x+x, & 0 < x \leq 1 \\ x+x-1, & x > 1 \end{cases}$
 $= \begin{cases} 2x-1, & x \leq 0 \\ 2x, & 0 < x \leq 1 \\ 2x-1, & x > 1 \end{cases} = \begin{cases} 2x-1, & x \leq 0 \text{ ή } x > 1 \\ 2x, & 0 < x \leq 1 \end{cases}$

ii) $D_{f \cdot g} = D_f \cap D_g = \mathbb{R}$ $(f \cdot g)(x) = f(x) \cdot g(x) = \begin{cases} x(x-1), & x \leq 0 \\ x \cdot x, & 0 < x \leq 1 \\ x(x-1), & x > 1 \end{cases}$
 $= \begin{cases} x^2-x, & x \leq 0 \text{ ή } x > 1 \\ x^2, & 0 < x \leq 1 \end{cases}$

α) $D_{\frac{f}{g}} = D_f \cap D_g - \{x / g(x) = 0\} = \mathbb{R} - \{0\} = \mathbb{R}^* = (-\infty, 0) \cup (0, +\infty)$

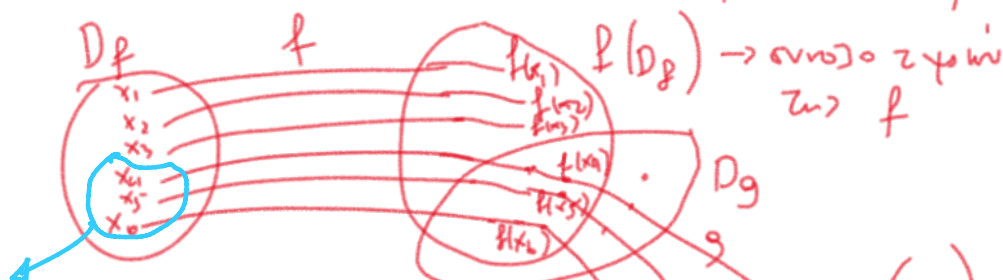
$g(x) = 0 \Leftrightarrow x = 0$

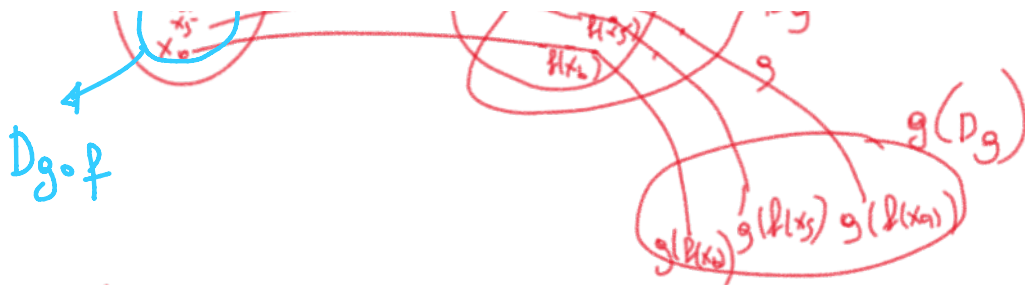
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \begin{cases} \frac{x-1}{x}, & x < 0 \\ \frac{x}{x}, & 0 < x \leq 1 \\ \frac{x}{x-1}, & x > 1 \end{cases} = \begin{cases} \frac{x-1}{x}, & x < 0 \\ 1, & 0 < x \leq 1 \\ \frac{x}{x-1}, & x > 1 \end{cases}$

Σύνθεση Συναρτήσεων

$f: D_f \rightarrow \mathbb{R}$ $g: D_g \rightarrow \mathbb{R}$

Σύνθεση $z \mapsto f$ με $z \in g$ (σύνθεση $g \circ f$)

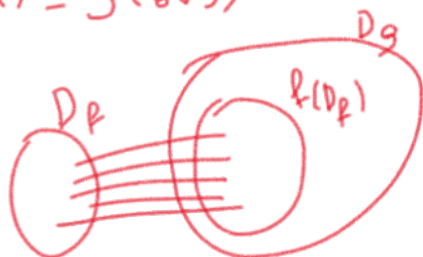




$$D_{g \circ f} = \{x \in D_f \text{ και } f(x) \in D_g\}$$

$$(g \circ f)(x) = g(f(x))$$

$$\text{Προφανώς } D_{g \circ f} \subseteq D_f$$



$$D_{g \circ f} = D_f \Leftrightarrow f(D_f) \subseteq D_g$$



$$f(D_f) \cap D_g = \emptyset \Leftrightarrow g \circ f \text{ δεν ορίζεται}$$

Σύνθεση $u \rightarrow g$ με $u \leftarrow f = f \circ g$

$$D_{f \circ g} = \{x \in D_g \text{ και } g(x) \in D_f\} \subseteq D_g$$

$$(f \circ g)(x) = f(g(x))$$

$$D_{f \circ g} = D_g \Leftrightarrow g(D_g) \subseteq D_f$$

$$f \circ g \text{ δεν ορίζεται} \Leftrightarrow g(D_g) \cap D_f = \emptyset$$

$$\text{Γενικά } f \circ g \neq g \circ f \quad \text{όπως } (g \circ f) \circ h = g \circ (f \circ h) = g \circ f \circ h$$

Θέμα 3ο (σύνθεση συναρτήσεων)

Να βρείτε τις συναρτήσεις $f \circ g$, $g \circ f$, $f \circ f$ στις παρακάτω περιπτώσεις:

$$\alpha) f(x) = \sqrt{x-1}, g(x) = \ln x \quad , \quad \beta) f(x) = \ln x, g(x) = \begin{cases} x, & x \leq 1 \\ x-1, & x > 1 \end{cases}$$

$$\alpha) D_f = [1, +\infty) \quad , \quad D_g = (0, +\infty)$$

$$D_{f \circ g} = \{x \in D_g \text{ και } g(x) \in D_f\} = \{x > 0 \text{ και } \ln x \geq 1\} = \{x > e\} = (e, +\infty)$$

$$= \{x > 0 \text{ dan } x \geq e\} = [e, +\infty)$$

$$(f \circ g)(x) = f(g(x)) = \sqrt{\ln x - 1}$$

$$D_{g \circ f} = \{x \in D_f \text{ dan } f(x) \in D_g\} = \{x \geq 1 \text{ dan } \sqrt{x-1} > 0\} = (1, +\infty)$$

$$(g \circ f)(x) = g(f(x)) = \ln f(x) = \ln \sqrt{x-1} = \frac{1}{2} \ln(x-1)$$

$$D_{f \circ f} = \{x \in D_f \text{ dan } f(x) \in D_f\} = \{x \geq 1 \text{ dan } \sqrt{x-1} \geq 1\}$$

$$= \{x \geq 1 \text{ dan } x-1 \geq 1\} = \{x \geq 1 \text{ dan } x \geq 2\} = [2, +\infty)$$

$$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)-1} = \sqrt{\sqrt{x-1}-1}$$