Μάθημα 91ο - Θεμελιώδες θεώρημα ολοκληρωτικού λογισμού

Θέμα 13ο Υπολογίστε τα ολοκληρώματα :

i) $\displaystyle\int_1^2 \frac{2x^2-5x+3}{2x^2}dx$ ii) $\displaystyle\int_{-1}^1 \sqrt[3]{x^2}dx$ iii) $\displaystyle\int_0^{\frac{\pi}{4}} \varepsilon\phi^2 x dx$ iv) $\displaystyle\int_0^1 (x^2e^x+2xe^x)dx$

v) $\displaystyle\int_1^2 \frac{xe^x-e^x}{x^2}dx$ vi) $\displaystyle\int_0^1 (2x+1)\sigma\upsilon\nu(x^2+x+3)dx$ vii) $\displaystyle\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}dx$ viii) $\displaystyle\int_0^1 \frac{1}{2e^{-x}+1}dx$

ix) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\eta\mu^2 x\cdot\sigma\upsilon\nu^2 x}dx$ x) $\displaystyle\int_{\frac{\pi}{9}}^{\frac{\pi}{3}} \frac{1}{\eta\mu x\,\sigma\upsilon\nu x}dx$ xi) $\displaystyle\int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^3 x\,dx$

xii) $\displaystyle\int_2^4 \frac{1}{x\ln x}$ xiii) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\varepsilon\phi^2 x}{\varepsilon\phi x}dx$

i) $$\int_1^2 \frac{2x^2-5x+3}{2x^2}dx = \int_1^2\left(\frac{2x^2}{2x^2}-\frac{5x}{2x^2}+\frac{3}{2x^2}\right)dx = \int_1^2\left(1-\frac{5}{2x}+\frac{3}{2x^2}\right)dx$$

$$=\left[x-\frac{5}{2}\ln x-\frac{3}{2x}\right]_1^2=\dots$$

ii) $$\int_{-1}^1 \sqrt[3]{x^2} = \int_{-1}^1 |x|^{\frac{2}{3}}dx = \int_{-1}^0(-x)^{\frac{2}{3}}dx+\int_0^1 x^{2/3}dx$$

$$=\left[-\frac{(-x)^{5/3}}{5/3}\right]_{-1}^0+\left[\frac{x^{5/3}}{5/3}\right]_0^1=\dots$$

iii) $$\int_0^{\frac{\pi}{4}}\varepsilon\phi^2 x\,dx = \int_0^{\frac{\pi}{4}}(\varepsilon\phi^2 x+1-1)dx = [\varepsilon\phi x - x]_0^{\pi/4} = \varepsilon\phi\frac{\pi}{4}-\frac{\pi}{4}=1-\frac{\pi}{4}$$

$$(\varepsilon\phi x)' = \frac{1}{\sigma\upsilon\nu^2 x} = 1+\varepsilon\phi^2 x$$

iv) $$\int_0^1(x^2e^x+2xe^x)dx = \left[x^2e^x\right]_0^1 = e$$

v) $$\int_1^2\frac{xe^x-e^x}{x^2}dx = \left[\frac{e^x}{x}\right]_1^2 = \frac{e^2}{2}-e$$

vi) $\int_0^1 (2x+1)\,\text{συν}(x^2+x+3)\,dx = \Big[\text{ημ}(x^2+x+3)\Big]_0^1 = \text{ημ}5 - \text{ημ}3$

vii) $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \Big[2\sqrt{\ln x}\Big]_e^{e^2} = 2\sqrt{\ln e^2} - 2\sqrt{\ln e} =$
$= 2\sqrt{2} - 2$

viii) $\int_0^1 \frac{1}{2e^{-x}+1}\,dx = \int_0^1 \frac{1}{\frac{2}{e^x}+1}\,dx = \int_0^1 \frac{1}{\frac{2+e^x}{e^x}}\,dx =$
$\int_0^1 \frac{e^x}{e^x+2}\,dx = \Big[\ln(e^x+2)\Big]_0^1 = \ln(e+2) - \ln 3$

ix) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\text{ημ}^2x\cdot\text{συν}^2x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\text{ημ}^2x+\text{συν}^2x}{\text{ημ}^2x\cdot\text{συν}^2x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{\text{ημ}^2x}{\text{ημ}^2x\,\text{συν}^2x} + \frac{\text{συν}^2x}{\text{ημ}^2x\,\text{συν}^2x}\right)dx$
$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{1}{\text{συν}^2x} + \frac{1}{\text{ημ}^2x}\right)dx = \Big[\text{εφ}x - \text{σφ}x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = ---$

x) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\text{ημ}x\cdot\text{συν}x} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\text{ημ}^2x+\text{συν}^2x}{\text{ημ}x\cdot\text{συν}x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{\text{ημ}^2x}{\text{ημ}x\,\text{συν}x} + \frac{\text{συν}^2x}{\text{ημ}x\cdot\text{συν}x}\right)dx$
$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{\text{ημ}x}{\text{συν}x} + \frac{\text{συν}x}{\text{ημ}x}\right)dx = \Big[-\ln\text{συν}x + \ln\text{ημ}x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \Big[\ln\frac{\text{ημ}x}{\text{συν}x}\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} =$
$= \Big[\ln\text{εφ}x\Big]_{\frac{\pi}{4}}^{\pi/3} = \ln\sqrt{3}$

xi) $\int_0^{\frac{\pi}{2}} \text{συν}^3x\,dx = \int_0^{\frac{\pi}{2}} \text{συν}x\cdot\text{συν}^2x\,dx = \int_0^{\frac{\pi}{2}} \text{συν}x(1-\text{ημ}^2x)\,dx =$
$= \int_0^{\frac{\pi}{2}}(\text{συν}x - \text{ημ}^2x\,\text{συν}x)\,dx = \Big[\text{ημ}x - \frac{\text{ημ}^3x}{3}\Big]_0^{\frac{\pi}{2}} = 1 - \frac{1}{3} = \frac{2}{3}$

$$= \int_0^{\frac{\pi}{2}} (\sigma\upsilon\nu x - \eta\mu^2 x\cdot\sigma\upsilon\nu x)\,dx = \left[\eta\mu x - \frac{\eta\mu^3 x}{3}\right]_0^{\frac{\pi}{2}} = 1 - \frac{1}{3} = \frac{2}{3}$$

xii) $\displaystyle\int_2^4 \frac{1}{x\ln x}\,dx = \left[\ln\ln x\right]_2^4 = \ln\ln 4 - \ln\ln 2 = \ln\frac{\ln 4}{\ln 2} =$

$$= \ln\frac{\ln 2^2}{\ln 2} = \ln\frac{2\ln 2}{\ln 2} = \ln 2$$

xiii) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\varepsilon\varphi^2 x}{\varepsilon\varphi x}\,dx = \left[\ln\varepsilon\varphi x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \ln\sqrt{3}$

$\dfrac{(\varepsilon\varphi x)'}{\varepsilon\varphi x}$

Άσκηση

Έστω $f:\mathbb{R}\to\mathbb{R}$ συνεχής για την οποία ισχύει $f(x) = x + \int_0^2 x f(x)\,dx$

Να βρεθεί ο τύπος της f.

Λύση

Έστω $\int_0^2 x f(x)\,dx = c$ όπου $c\in\mathbb{R}$ οπότε $f(x) = x + c$

Αναζητώ το c

Είναι $c = \int_0^2 x f(x)\,dx = \int_0^2 x(x+c)\,dx = \int_0^2 (x^2 + cx)\,dx = \left[\frac{x^3}{3} + \frac{cx^2}{2}\right]_0^2 =$

$\frac{8}{3} + 2c$

Άρα $c = \frac{8}{3} + 2c \Rightarrow \boxed{c = -\frac{8}{3}}$

Τελικά $\boxed{f(x) = x - \frac{8}{3}}$