12. Έστω οι συναρτήσεις f, g, με f'', g'' συνεχείς στο $[\alpha, \beta]$. Αν $f(\alpha)=g(\alpha)=0$ και $f'(\beta)=g'(\beta)$, να αποδείξετε ότι

$$I=\int_\alpha^\beta \big(f(x)g''(x)-f''(x)g(x)\big)dx = g'(\beta)(f(\beta)-g(\beta)).$$

$$I=\int_a^b f(x)g''(x)dx-\int_a^b f''(x)g(x)dx=\int_a^b f(x)\cdot(g'(x))'dx-\int_a^b (f'(x))'g(x)dx=$$

$$=\big[f(x)g'(x)\big]_a^b-\int_a^b f'(x)\,g'(x)dx-\left(\big[f'(x)g(x)\big]_a^b-\int_a^b f'(x)\,g'(x)\,dx\right)=$$

$$=f(b)g'(b)-f(a)g'(a)-\cancel{\int_a^b f'(x)g'(x)\,dx}-f'(b)g(b)+f'(a)g(a)+\cancel{\int_a^b f'(x)g'(x)\,dx}=$$

$$=f(b)\cdot g'(b)-0-g'(b)\cdot g(b)+0=g'(b)\big(f(b)-g(b)\big)$$

4. Αν $I_\nu=\int_0^1 \frac{t^{2\nu+1}}{1+t^2}dt,\quad \nu\in\mathbb{N}$,

i) Να υπολογίσετε το άθροισμα $I_\nu+I_{\nu+1}$, $\nu\in\mathbb{N}$

ii) Να υπολογίσετε τα ολοκληρώματα I_0, I_1, I_2.

i) $$I_\nu+I_{\nu+1}=\int_0^1\frac{t^{2\nu+1}}{1+t^2}dt+\int_0^1\frac{t^{2(\nu+1)+1}}{1+t^2}dt=$$

$$=\int_0^1\frac{t^{2\nu+1}}{1+t^2}dt+\int_0^1\frac{t^{2\nu+3}}{1+t^2}dt=\int_0^1\frac{t^{2\nu+1}+t^{2\nu+3}}{1+t^2}dt=\int_0^1\frac{t^{2\nu}(1+t^2)}{1+t^2}dt$$

$$=\int_0^1 t^{2\nu}dt=\left[\frac{t^{2\nu+1}}{2\nu+1}\right]_0^1=\frac{1}{2\nu+1}$$

ii) $$I_0=\int_0^1\frac{t}{1+t^2}dt=\left[\frac{\ln(1+t^2)}{2}\right]_0^1=\frac{\ln 2}{2}$$

$$I_\nu+I_{\nu+1}=\frac{1}{2\nu+1}$$

για $\nu=0$: $I_0+I_1=1 \Rightarrow I_1=1-I_0 \Rightarrow \boxed{I_1=1-\frac{\ln 2}{2}}$

για $\nu=1$: $I_1+I_2=\frac{1}{3} \Rightarrow I_2=\frac{1}{3}-I_1 \Rightarrow I_2=\frac{1}{3}-1+\frac{\ln 2}{2} \Rightarrow \boxed{I_2=\frac{\ln 2}{2}-\frac{2}{3}}$

i) Να υπολογίσετε το ολοκλήρωμα $\int_0^{1/2}\frac{1}{x^2-1}dx \overset{(*)}{=} \int_0^{\frac{1}{2}}\left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx=$

$$=\left[-\frac{1}{2}\ln(x+1)+\frac{1}{2}\ln|x-1|\right]_0^{\frac{1}{2}}=\ldots$$

(*)

$$\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$$

Είναι $$\frac{\overset{x-1}{A}}{x+1}+\frac{\overset{x+1}{B}}{x-1}=\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}=\frac{Ax-A+Bx+B}{(x+1)(x-1)}=\frac{(A+B)x+B-A}{(x+1)(x-1)}$$

Είναι $\frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} = \frac{Ax-A+Bx+B}{(x+1)(x-1)} = \frac{(A+B)x+B-A}{(x+1)(x-1)}$

Πρέπει $(A+B)x + B - A = 1$, για κάθε x

$\left.\begin{array}{l} A+B=0 \\ B-A=1 \end{array}\right| \xrightarrow{(+)} 2B=1 \Rightarrow B=\frac{1}{2}$ και $A=-\frac{1}{2}$

iii) $\int_0^1 x\ln(9+x^2)dx = \int_0^1 \left(\frac{x^2}{2}\right)' \ln(9+x^2)\,dx = \left[\frac{x^2}{2}\ln(9+x^2)\right]_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{2x}{9+x^2}dx$

$= \frac{1}{2}\ln 10 - \int_0^1 \frac{x^3}{x^2+9}dx = \frac{1}{2}\ln 10 - \int_0^1\left(x - \frac{9x}{x^2+9}\right)dx =$

$= \frac{1}{2}\ln 10 - \left[\frac{x^2}{2} - \frac{9}{2}\ln(x^2+9)\right]_0^1$

$= \dots$

$$\begin{array}{r|l} x^3 & x^2+9 \\ \cline{2-2} -x^3-9x & x \\ \hline -9x & \end{array}$$

$x^3 = x\cdot(x^2+9) - 9x$

$\frac{x^3}{x^2+9} = x - \frac{9x}{x^2+9}$

Άσκηση

Έστω $F(x)$ μια παράγουσα της $f(x) = e^{x^2}$ με $F(1)=0$.

Να υπολογίσετε το $\int_0^1 F(x)dx$.

Λύση

$\int_0^1 F(x)dx = \int_0^1 (x)' F(x)dx = \left[xF(x)\right]_0^1 - \int_0^1 x F'(x)dx = F(1) - 0 - \int_0^1 x f(x)dx$

$= 0 - \int_0^1 x\cdot e^{x^2}dx = -\left[\frac{e^{x^2}}{2}\right]_0^1 = -\left(\frac{e}{2} - \frac{1}{2}\right) = \frac{1-e}{2}$

Θέμα 16ο

Έστω συνάρτηση f 2 φορές παραγωγίσιμη με συνεχή $2^η$ παράγωγο στο $[\alpha, \beta]$ για την οποία ισχύει ότι : $\int_\alpha^\beta x f''(x)dx = 0$. Να δείξετε ότι :

Οι εφαπτομένες της γραφικής παράστασης της f στα σημεία με τετμημένες α και β τέμνονται πάνω στον ψψ΄

$\int_a^b x\left(f'(x)\right)' = 0 \Longrightarrow \left[x f'(x)\right]_a^b - \int_a^b (x)' f'(x)dx = 0 \Longrightarrow$

$b\cdot f'(b) - a f'(a) - \int_a^b f'(x)dx = 0 \Longrightarrow b f'(b) - a f'(a) - \left[f(x)\right]_a^b = 0 \Rightarrow$

$b f'(b) - a f'(a) - f(b) + f(a) = 0 \longrightarrow \boxed{b f'(b) - f(b) = a f'(a) - f(a)}$ (1)

$\varepsilon_a: y - f(a) = f'(a)(x-a)$ για $x=0 \to y_a = f(a) - a f'(a)$ (2)

$\varepsilon_b: y - f(b) = f'(b)(x-b)$ για $x=0 \to y_b = f(b) - b f'(b)$ (3)

(1), (2), (3) $\Longrightarrow y_a = y_b$ οπότε $\varepsilon_a, \varepsilon_b$ τέμνονται στον yy΄

$$\int_\alpha^\beta f(g(x))g'(x)dx = \int_{u_1}^{u_2} f(u)du,$$

όπου f, g' είναι συνεχείς συναρτήσεις, $u = g(x)$, $du = g'(x)dx$ και $u_1 = g(\alpha)$, $u_2 = g(\beta)$.

7. Να υπολογίσετε τα ολοκληρώματα

i) $\displaystyle\int_4^6 \frac{x}{\sqrt{x^2-4}}dx$ ii) $\displaystyle\int_0^{\pi/2}[\eta\mu(\sigma\upsilon\nu x + x)\eta\mu x - \eta\mu(\sigma\upsilon\nu x + x)]dx$.

iii) $\displaystyle\int_0^1 x^2\sqrt{x+1}\,dx$ iv) $\displaystyle\int_0^{\ln 2} \frac{1}{\sqrt{1+e^x}}dx$ v) $\displaystyle\int_1^{64} \frac{1}{2\sqrt{x}\cdot\sqrt[3]{x}}dx$ vi) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\eta\mu x}{\eta\mu^2 x + \sigma\upsilon\nu x + 1}dx$

√ **ii) Να υπολογίσετε το ολοκλήρωμα** $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{\eta\mu x}dx$.

i) $\displaystyle\int_4^6 \frac{x}{\sqrt{x^2-4}}dx = \int_{12}^{32} \frac{1/2}{\sqrt{u}}du = \int_{12}^{32} \frac{1}{2\sqrt{u}}du = \left[\sqrt{u}\right]_{12}^{32} = \sqrt{32} - \sqrt{12} =$

$= 4\sqrt{2} - 2\sqrt{2} = 2\sqrt{2}$

Θέτω $x^2 - 4 = u$ | για $x = 4 \to u = 12$

$2x\,dx = 1\,du$ | για $x = 6 \to u = 32$

$x\,dx = \frac{1}{2}du$

ii) $\displaystyle\int_0^{\pi/2}[\eta\mu(\sigma\upsilon\nu x + x)\eta\mu x - \eta\mu(\sigma\upsilon\nu x + x)]dx = \int_0^{\frac{\pi}{2}} \eta\mu(\sigma\upsilon\nu x + x)\cdot(\eta\mu x - 1)\,dx =$

$\displaystyle = \int_1^{\frac{\pi}{2}} \eta\mu u\,(-du) = \left[\sigma\upsilon\nu u\right]_1^{\frac{\pi}{2}} =$

$- \sigma\upsilon\nu 1$

Θέτω $\sigma\upsilon\nu x + x = u$ | $x = 0 \to u = 1$

$(-\eta\mu x + 1)dx = du$ | $x = \frac{\pi}{2} \to u = \frac{\pi}{2}$

iii) $\displaystyle\int_0^1 x^2\sqrt{x+1}\,dx = \int_1^2 (u-1)^2\cdot\sqrt{u}\,du = \int_1^2 (u^2 - 2u + 1)\cdot u^{1/2}du = \int_1^2\left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$

Θέτω $x + 1 = u$ | $x = 0 \to u = 1$

$dx = du$ | $x = 1 \to u = 2$

$\displaystyle = \left[\frac{u^{7/2}}{7/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right]_1^2 = \cdots$

iv) $\displaystyle\int_0^{\ln 2} \frac{1}{\sqrt{1+e^x}}dx = \int_0^{\ln 2} \frac{e^x}{\sqrt{1+e^x}\cdot e^x}dx = \int_2^3 \frac{2u}{u\cdot(u^2-1)}du =$

Θέτω $\sqrt{1+e^x} = u \Rightarrow e^x = u^2 - 1$ | $x = 0 \Rightarrow u = 2$

$e^x dx = 2u\,du$ | $x = \ln 2 \Rightarrow u = 3$

$\displaystyle = \int_2^3 \frac{2}{u^2-1}du \overset{*}{=}$

$$\frac{2}{u^2-1} = \frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{A(u-1)+B(u+1)}{(u+1)(u-1)} = \frac{(A+B)u+B-A}{(u+1)(u-1)}$$

$$\left.\begin{array}{l} A+B=0 \\ B-A=2 \end{array}\right| \Rightarrow \begin{array}{l} A=-1 \\ B=1 \end{array}$$

$$(*) = \int_2^3 \left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = \Big[-\ln(u+1) + \ln(u-1)\Big]_2^3 = \ldots$$

v) $$\int_1^{64} \frac{1}{2\sqrt{x} - \sqrt[3]{x}}\,dx = \int_1^2 \frac{6u^5}{2u^3-u^2}\,du = \int_1^2 \frac{6u^5}{u^2(2u-1)}\,du = \int_1^2 \frac{6u^3}{2u-1}\,du \quad (*)$$

Θέτω $\sqrt[6]{x} = u \Rightarrow x = u^6$, $dx = 6u^5\,du$ | $x=1 \Rightarrow u=1$, $x=64 \Rightarrow u = \sqrt[6]{64} = 2$ | $\sqrt{x} = \sqrt{u^6} = u^{6/2} = u^3$, $\sqrt[3]{x} = \sqrt[3]{u^6} = u^{6/3} = u^2$

$$\underline{\underline{(*)}} \int_1^2 \left(3u^2 + \frac{3}{2}u + \frac{3}{4} + \frac{3/4}{2u-1}\right)du =$$

$$= \left[u^3 + \frac{3}{4}u^2 + \frac{3}{4}u + \frac{3}{8}\ln(2u-1)\right]_1^2 = \ldots$$

$$\begin{array}{l|l} 6u^3 & 2u-1 \\ \hline -6u^3+3u^2 & 3u^2+\frac{3}{2}u+\frac{3}{4} \\ 3u^2 & \\ -3u^2+\frac{3}{2}u & \\ \frac{3}{2}u & \\ -\frac{3}{2}u+\frac{3}{4} & \\ \frac{3}{4} & \end{array}$$

vi) $$\int_0^{\frac{\pi}{2}} \frac{\eta\mu x}{\eta\mu^2 x + \sigma\upsilon\nu x + 1}\,dx = \int_1^0 \frac{-1}{1-u^2+u+1}\,du = \int_1^0 \frac{1}{u^2-u-2}\,du$$

Θέτω $\sigma\upsilon\nu x = u$, $-\eta\mu x\,dx = du$ | $x=0 \to u=1$, $x=\frac{\pi}{2} \to u=0$ | $\eta\mu^2 x = 1-\sigma\upsilon\nu^2 x = 1-u^2$

$$\frac{1}{u^2-u-2} = \frac{1}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1} = \frac{(A+B)u+A-2B}{(u-2)(u+1)}$$

$$\left.\begin{array}{l} A+B=0 \\ A-2B=1 \end{array}\right| \begin{array}{l} B=\frac{1}{3} \\ A=-\frac{1}{3} \end{array}$$

$$\underline{\underline{*}} \int_1^0 \left(\frac{-1/3}{u-2} + \frac{1/3}{u+1}\right)du = \left[-\frac{1}{3}\ln|u-2| + \frac{1}{3}\ln|u+1|\right]_1^0 = \ldots$$

ii) Να υπολογίσετε το ολοκλήρωμα $\int_{\pi/3}^{\pi/2} \frac{1}{\eta\mu x}\,dx$. $= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\eta\mu x}{\eta\mu^2 x}\,dx =$

$\int^{\frac{\pi}{2}}$ $\int^{\frac{1}{2}}$ $\int^{1}$

$$\int_{\pi/3} \eta\mu x \qquad \int_{\frac{\pi}{3}} \frac{\quad}{\eta\mu^2 x}\,dx =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\eta\mu x}{1-\sigma\upsilon\nu^2 x}\,dx \quad \int_0^{\frac{1}{2}} \frac{-1}{1-u^2}\,du = \int_0^{\frac{1}{2}} \frac{1}{u^2-1}\,du =$$

Θέτω $\sigma\upsilon\nu x = u$ | $x = \frac{\pi}{3} \to u = \frac{1}{2}$

$-\eta\mu x\,dx = du$ | $x = \frac{\pi}{2} \to u = 0$

$$= \int_0^{\frac{1}{2}} \left(\frac{-1/2}{u+1} + \frac{1/2}{u-1} \right) du$$

$$= \left[-\frac{1}{2}\ln(u+1) + \frac{1}{2}\ln|u-1| \right]_0^{\frac{1}{2}} = \ldots$$